TRACK ELEMENT MERGING STRATEGY AND VERTEX FITTING IN COMPLEX MODULAR DETECTORS

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1. Introduction

With the construction of storage rings a new era of experimental high energy physics had begun. In the CERN intersecting storage rings center-of-mass energies have been reached which have not been obtainable by fixed-target experiments so far. At these high energies particle multiplicities are high and detectors are complex.

At the same time, progress in electronics allowed the acquisition of events at high rates, and high-resolution detectors have been built which achieve the precision necessary to measure the momentum of energetic particles.

The analysis of the data requires that the reconstruction algorithms must be of a precision at least equivalent to that of the measurements, and must be sufficiently fast to be able to deal with the amount of collected data. Furthermore, they must be flexible so that the reconstruction program is able to deal with any required combination of individual detector modules, but they must still be of a simple architecture. Therefore, to make the mathematical algorithms fully efficient, they must be conceived in close connection with the design of the hardware components.

For the purpose of this paper it is assumed that the problem of associating the many hundreds and often ambiguous coordinates to particle track candidates has already been solved. Then it turns out that the requirements of track fitting, i.e. estimation of track parameters, are met best by the least-squares estimate, being rather fast, flexible and relatively robust. If this method is to be optimal, the weight matrix must vary sufficiently smoothly as a function of the track parameters in the neighbourhood of an individual track's path, and its covariant (off-diagonal) terms must be well understood.

The first condition is well fulfilled for multiwire proportional chambers, but must be carefully checked for novel-type detectors like drift tubes working in the streamer mode. The second condition is rather easily fulfilled as far as multiple scattering is concerned, but might be more complicated in case of a dense stack of wire or microstrip detectors. If, in addition, the track model can be approximated sufficiently well by a linear model in the neighbourhood of the measurements belonging to an individual track, the least-squares estimator has minimum variance among all linear estimators. Asymptotically, or in case of Gaussian measurement errors, this estimator is also efficient. The linear properties of the least-squares method and the equivalence of its mean quadratic properties with those of a Gaussian process make this method also well adapted for error propagation and therefore for a subsequent fit of a vertex or a kinematical hypothesis, and also for a chi-square test.

In practice, it turns out that a tree-like hierarchical strategy is the best one to assemble coordinates or track elements from many particles to a common vertex. Therefore sect. 2 will discuss different attempts to

assemble individual track elements to complete tracks in the context of a complex detector with possibly radically different modules. It should clarify the precise meaning, advantages and disadvantages of all methods discussed. Sect. 3 will describe a novel algorithm to evaluate the common vertex in the presence of high track multiplicities. Its main advantage is the fact that the number of required arithmetic operations is proportional to n (number of tracks) if only the estimate and the χ^2 are computed, and proportional to n^2 if the covariance matrix of the estimate is computed as well.

The guiding line for this report was a case study (the LEP experiment DELPHI at CERN), but all methods discussed are of general use and can be applied without any restriction to any similar experiment.

2. Track element merging strategy

2.1. Generalities

The path of a charged particle in a magnetic field is given by the solution of a differential equation. Neglecting energy loss and the effect of an electric field, the absolute value of the momentum, $|P|$, remains unchanged in vacuum, and is a constant of motion [1,2]. In this case the equation of motion can be written in the simple form:

$$d^2r/ds^2 = (q/|P|)(dr/ds \times B(r)),\qquad(1a)$$

where
r is the Cartesian coordinate vector (x, y, z),
s is the path length, with $ds^2 = dx^2 + dy^2 + dz^2$,
q is a constant proportional to the charge [1],
B is the magnetic induction.

In the presence of matter along the particle's path, eq. (1a) has to be modified:

$$d^2r/ds^2 = (q/|P|)(dr/ds \times B(r)) + \vartheta(s),\qquad(1b)$$

where $\vartheta(s)$ is a "white noise"-like stochastic process which describes multiple scattering. $|P|$ is no longer a constant, but a falling function of s:

$$|P(s)| = |P(0)| \cdot w(s),$$

$$w(s) = 1, \quad \text{for} \quad s = 0,$$

$$w(s_2) \leqslant w(s_1), \qquad \text{for} \quad s_2 > s_1.$$

Unlike $\vartheta(s)$, $w(s)$ can be considered as a deterministic function.

Frequently it is convenient to choose one of the coordinates, e.g. z, as the independent variable instead of s. The preferred choice is the axis perpendicular to the detector surfaces.

The position of the undisturbed track at a certain value of z is then a function of z and of the set of initial conditions p_R at a reference plane $z = z_R$:

$$x(z) = f_x(z, p_R), \quad y(z) = f_y(z, p_R).\qquad(2a)$$

p_R is a five-dimensional vector:

$$p_R^1 = (x_R, y_R, (dx/dz)_R, (dy/dz)_R, 1/|P_R|).\qquad(2b)$$

If one takes into account multiple scattering, the solution of the equation of motion can be written as

$$x(z) = f_x(z, p_R) + \epsilon_x^{ms}(z), \quad y(z) = f_y(z, p_R) + \epsilon_y^{ms}(z),\qquad(3a)$$

provided that B is sufficiently homogeneous over a range covered by the ensemble of scattered tracks with
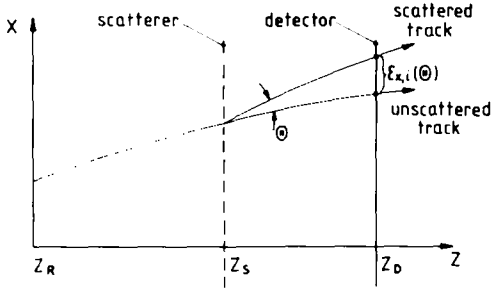
Fig. 1. Deviation of the actual (scattered) track from an ideal (unscattered) track at a detector plane at $z = z_D$.

the same initial conditions. The random processes ϵ_x^{ms} and ϵ_y^{ms} describe the particle's deviation from the unscattered track (fig. 1).

By introducing

$$\frac{dx}{dz} = \frac{df_x}{dz} + \frac{d\epsilon_x^{ms}}{dz}, \quad \frac{dy}{dz} = \frac{df_y}{dz} + \frac{d\epsilon_y^{ms}}{dz} \tag{3b}$$

and

$$1/|P_z| = 1/(|P_R| \cdot w'(z)),$$

one can define a vector p_z similar to p_R (see eq. (2b)), which describes the track behaviour at any value of z. Note that z_R is not necessarily the physical starting point of the particle's path, which may also extend to $z < z_R$. It is, however, essential to understand that ϵ_x^{ms}, ϵ_y^{ms} and their derivatives are set to 0 at $z = z_R$ by definition, and that, in general, different solutions of the equation of motion will be selected for different choices of z_R (fig. 2).

Assuming absence of multiple scattering, the parameter vector p_z is a deterministic function of p_R:

$$p_z = p(p_R; z). \tag{4a}$$

The function p is called the track model. For small variations Δp_R of the reference parameters p_R one can write:

$$p_z(p_R + \Delta p_R) = p_z(p_R) + \frac{\partial p_z}{\partial p_R} \cdot \Delta p_R + o(\Delta p_R), \tag{4b}$$

where $\partial p_z/\partial p_R$ propagates a change in the parameters at $z = z_R$ to the point z. More generally, one can
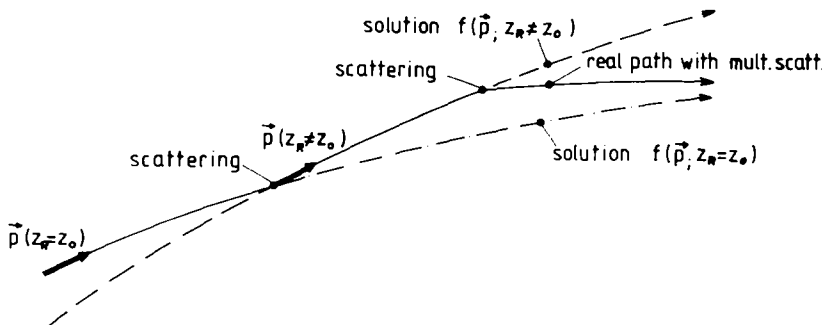

Fig. 2. Track with discrete multiple scattering; for different choices of z_R different solutions of the equation of motion are selected.

propagate a variation of the parameters at $z = z_i$ to $z = z_k$ by means of the formula [3–5]:

$$\frac{\partial p_{z_k}}{\partial p_{z_i}} = \frac{\partial p_{z_k}}{\partial p_R} \cdot \frac{\partial p_R}{\partial p_{z_i}} = \frac{\partial p_{z_k}}{\partial p_R} \left(\frac{\partial p_{z_i}}{\partial p_R} \right)^{-1} . \tag{4c}$$

In practice, the track model p and the "differential propagator" $\partial p_{z_k}/\partial p_{z_i}$ can be obtained either
– by an explicit solution of eq. (1a);
– or by numerical integration of a reference track and five auxiliary tracks, each of them varying one of the five reference parameters, i.e. numerical differentiation with respect to the reference parameters;
    or by other approximation methods [6–8].
    In general, a suitable choice of track parameters can be made such that small variations around the reference track are well described by a linear track model. If, however, large deviations from the reference track occur during the fitting procedure, it is more convenient to compute a new reference track, instead of computing eq. (4b) up to second order. If an explicit solution of eq. (1a) is known, the construction of a linear track model is in principle unnecessary, but may simplify the least-squares estimation (global method).
    If multiple scattering becomes important, and if B or the material traversed by the particle is rather inhomogeneous, the ansatz (3a) and the linear approximation as indicated in eq. (4b) may turn out to be inadequate. In this case, a composite model offers a way out [6]: In order to allow the model to stay close to the physical path of the scattered particle, break points are added to the track model, e.g. at the beam tube, at thick detectors or walls of vacuum vessels, or at arbitrary points inside massive obstacles. The track model is no longer a function of only five parameters, but two parameters (scattering angles) per break point have to be added. These additional parameters correspond to direct measurements of the scattering angles at certain points each with datum 0 and with an error given by the theory [9–11].
    The third method described in this section, the progressive method, makes use of the properties of both the global and the break point method [12,13]. First a partial track segment is fitted by the global method, the reference plane being between this segment and the rest of the track. After reversing the track direction at the reference plane, a local track model is calculated by using a track propagator similar to eq. (4c), extending as far as the next detector surface which is then taken as the new reference plane. In this way, detector after detector is added. Each detector can either supply simple coordinates, or more elaborate information such as track position and direction summarizing the output of a whole detector module (fig. 3).
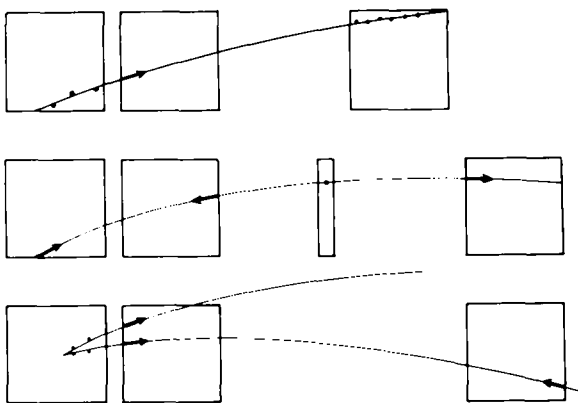

Fig. 3. Intermediate status of track fitting in a complex but modular detector.

2.2. The global method

According to the Gauss–Markov theorem, for a linear model the least-squares estimator has the smallest variance among all unbiased linear estimators, provided that the weight matrix used to compute the χ^2 is the inverse of the covariance matrix of the measurement errors. (Unbiased measurements and a nonsingular covariance matrix are assumed.)

With eq. (3a) and using the linear term of eq. (4b), the following least-squares ansatz can be formulated:

$$\mathcal{M}(\boldsymbol{p}_R) = \left(\boldsymbol{f}(\boldsymbol{p}_0) + \mathbf{A} \cdot (\boldsymbol{p}_R - \boldsymbol{p}_0) - \boldsymbol{m} \right)^T \cdot \mathbf{V}^{-1} \cdot \left(\boldsymbol{f}(\boldsymbol{p}_0) + \mathbf{A} \cdot (\boldsymbol{p}_R - \boldsymbol{p}_0) - \boldsymbol{m} \right), \tag{5a}$$

with:

$\boldsymbol{m} = (m_i)$, vector of measurements,
$\boldsymbol{f} = (f_i)$, vector of functions corresponding to $\boldsymbol{m}$,
$\mathbf{V}$, the covariance matrix of $\boldsymbol{m}$,
$\boldsymbol{p}_0$, the approximate initial value of the track at $z = z_R$,
$\mathbf{A} = \partial \boldsymbol{f} / \partial \boldsymbol{p}_R$ at the point $\boldsymbol{p}_0$.

The solution of the least-squares problem is then given by:

$$\tilde{\boldsymbol{p}}_R = \boldsymbol{p}_0 + (\mathbf{A}^T \mathbf{V}^{-1} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{V}^{-1} (\boldsymbol{m} - \boldsymbol{f}(\boldsymbol{p}_0)). \tag{5b}$$

The covariance matrix $\mathbf{V}$ is the sum of two independent contributions:

$$(\mathbf{V})_{ij} = \delta_{ij} \mathcal{E}\left\{ \epsilon_i^{\text{det}} \cdot \epsilon_j^{\text{det}} \right\} + \mathcal{E}\left\{ \epsilon_i^{\text{ms}} \cdot \epsilon_j^{\text{ms}} \right\}, \tag{6}$$

where ϵ_i^{det} is the intrinsic measurement error of the detecting device, i.e. the difference between the impact of the actual track and the measurement, whereas ϵ_i^{ms} is the deviation of the actual (scattered) track from the ideal (unscattered) track with the initial values $\boldsymbol{p}_0$. $\mathcal{E}$ denotes an expectation value.

For the evaluation of the matrix $\mathbf{V}$ it is convenient to distinguish between discrete scatterers (DS), where all material is concentrated into a single surface, and continuous media (CM). Then ϵ_i^{ms} can be approximated by:

$$\epsilon_i^{\text{ms}} = \sum_k \left(\frac{\partial f_i}{\partial \Theta_{1,k}} \Theta_{1,k} + \frac{\partial f_i}{\partial \Theta_{2,k}} \Theta_{2,k} \right) + \int \left(\frac{\partial f_i}{\partial \Theta_1(s)} \vartheta_1(s) + \frac{\partial f_i}{\partial \Theta_2(s)} \vartheta_2(s) \right) ds, \tag{7a}$$

with:

$\Theta_1(s)$, $\Theta_2(s)$, two independent scattering angles,
$\Theta_{i,k} = \Theta_i(s_k)$, the scattering angle at DS k,
$\vartheta_i(s) = d\Theta_i(s)/ds$,
s, the path length along the track.

The sum extends over all DS with $s_k < s_i$, the integral extends over all CM with $s < s_i$.

Using eq. (7a) one can compute

$$\mathcal{E}\left\{ \epsilon_i^{\text{ms}} \cdot \epsilon_j^{\text{ms}} \right\} = \sum_k \left(\frac{\partial f_i}{\partial \Theta_{1,k}} \frac{\partial f_j}{\partial \Theta_{1,k}} \mathcal{E}\left\{ \Theta_{1,k}^2 \right\} + \frac{\partial f_i}{\partial \Theta_{2,k}} \frac{\partial f_j}{\partial \Theta_{2,k}} \mathcal{E}\left\{ \theta_{2,k}^2 \right\} \right)$$

$$+ \int \left(\frac{\partial f_i}{\partial \Theta_1(s)} \frac{\partial f_j}{\partial \Theta_1(s)} \mathcal{E}\left\{ \vartheta_1^2(s) \right\} + \frac{\partial f_i}{\partial \Theta_2(s)} \frac{\partial f_j}{\partial \Theta_2(s)} \mathcal{E}\left\{ \vartheta_2^2(s) \right\} \right) ds, \tag{7b}$$

with $\mathcal{E}\{ \vartheta^2(s) \}$ the variance of multiple scattering angle per unit length.

In the derivation of eq. (7b) the independence of scattering angles at different points was used, i.e.:

$$\mathcal{E}\left\{ \vartheta(s_1) \vartheta(s_2) \right\} ds_1 ds_2 = \mathcal{E}\left\{ \vartheta^2(s_1) \right\} \delta(s_2 - s_1) ds_1 ds_2. \tag{7c}$$

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│   Master    │
│  Detector   │
│   ~→  → f(1) │
│   p_R ← p    │
└─────────────┘
      │
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┌──────────┐        ┌─────────┐
│ Updating │◄───────│ I (D_i1)│
│ ~→  → f(2)│        │   →     │
│ p_R ← p   │        │  m_i1   │
└──────────┘        └─────────┘
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┌──────────┐        ┌─────────┐
│ Updating │◄───────│ I (D_i2)│
│ ~→  → f(3)│        │   →     │
│ p_R ← p   │        │  m_i2   │
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│ Updating │◄───────│ I(D_i3 )│
│ ~→  → f(4)│        │   →     │
│ p_R ← p   │        │  m_i3   │
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$\vec{p}$ Masterparameter

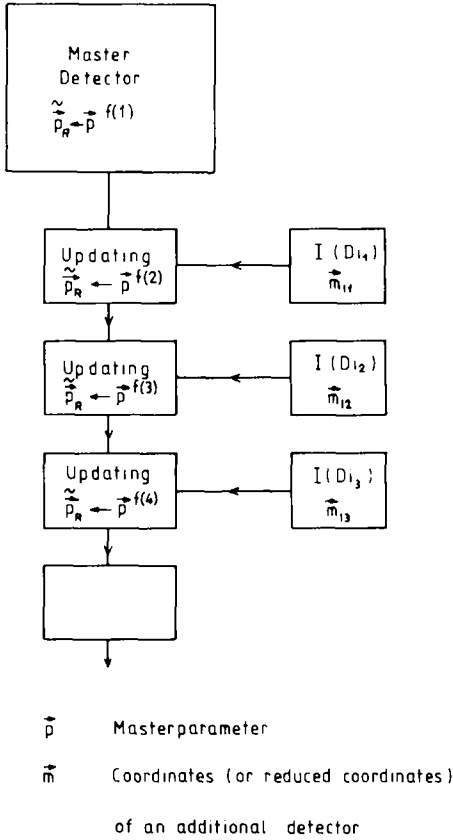$\vec{m}$ Coordinates (or reduced coordinates)

        of an additional detector

Fig. 4. The global method. Additional information in subsequent program steps is always used to improve the estimate of the same quantity, namely p_R (see also subsect. 2.4).

For efficient algorithms to evaluate eq. (7b), also including energy loss, see refs. [5,6]. Note that the term $\mathscr{E}\{\epsilon_l^{ms} \cdot \epsilon_l^{ms}\}$ must be evaluated for at most two independent coordinates per detector plane; additional coordinates can be handled by error propagation.

Due to the influence of multiple scattering, the actual track may stray quite far from the ideal, unscattered track. It should be stressed that the goal of the global method is a good estimation of the track parameters at the point where the particle crosses the reference plane. This is also important for error propagation and for comparison with the two other methods discussed below.

When information from additional detectors is added, the fitted value p_R (eq. (5b)) is in general correlated to the coordinates obtained from this additional detector. Using eq. (5b) one obtains:

$$\mathrm{cov}\left(\tilde{p}_R, m^{(add)}\right) = \left(A^T V^{-1} A\right)^{-1} A^T V^{-1} \cdot \mathrm{cov}\left(m, m^{(add)}\right), \tag{7d}$$

and $\mathrm{cov}(m_l, m_{l'}^{(add)})$ is calculated with eq. (7b); the prime denotes the index of additional coordinates.

Some advantages of this method are:
- transparent χ^2 ansatz;
- the asymptotic properties of the estimate are known;
- efficient algorithms are available;
- good starting point for a vertex fit;
   initialization is not critical;
- information from other detectors which may be added in subsequent program steps is always used to improve the estimate of the same quantity, namely p_R (fig. 4);

– after elimination of an outlying coordinate k, the inversion of the reduced covariance matrix can be avoided by updating the reduced weight matrix according to the following formula:

$$(\mathbf{V}^{-1})'_{i,j} = V_{ij}^{-1} - V_{ik}^{-1} \cdot V_{kj}^{-1} / V_{kk}^{-1}.$$

Some drawbacks are:
– The fitted track is an "extrapolated ideal track" and can be quite far from the physical track.
– The pull quantities (differences between measured and estimated impact on a detector) can be dominated by multiple scattering and therefore bear little information about the detector behaviour. This fact can also impede the judgement of possible outliers.
– A matrix of the dimension of the coordinate vector must be inverted. However, an important gain in computing time can be achieved if it is possible to split the covariance matrix into two independent projections. This is often the case for tracks with high momenta.

2.3. The break point method

This method is an attempt to follow the path of the scattered particle more closely. When the particle crosses a layer of material, two independent scattering angles are introduced which are considered as a direct measurement with datum 0 and an rms which is given by the Molière formula [14]. The actual values of the scattering angles are introduced into the fit as parameters to be estimated along with the standard track parameters.

If there are m breakpoints, the track model leads to functions of $5 + 2m$ parameters, in analogy to eq. (3a):

$$m = f(p_R, \mathbf{\Theta}) + \epsilon^{\text{det}},\tag{8}$$

with:

$\mathbf{\Theta} = (\Theta_{1,1}, \Theta_{2,1}, \ldots, \Theta_{1,m}, \Theta_{2,m})$, scattering angles, ϵ^{det} intrinsic measurement errors of the detectors.

The least-squares ansatz is now a sum of two terms, the first dealing with the detector errors and the second with the scattering at the break points (see also eq. (5a)):

$$\mathcal{M}(p_R, \mathbf{\Theta}) = \mathcal{M}_1 + \mathcal{M}_2,\tag{9a}$$

$$\mathcal{M}_1(p_R, \mathbf{\Theta}) = (f(p_R, \mathbf{\Theta}) - m)^T \mathbf{V}^{-1} (f(p_R, \mathbf{\Theta}) - m),\tag{9b}$$

$$\mathcal{M}_2(p_R, \mathbf{\Theta}) = \mathbf{\Theta}^T \mathbf{S}^{-1} \mathbf{\Theta},\tag{9c}$$

with $\mathbf{V}$ the covariance matrix of detector errors (usually diagonal) and $\mathbf{S}$ the covariance matrix of scattering angles (diagonal).

Again, a linear expansion around an approximate initial value p_0 is performed:

$$f(p_R, \mathbf{\Theta}) = f(p_0, 0) + \mathbf{A} \cdot (p_R - p_0) + \mathbf{F} \cdot \mathbf{\Theta},\tag{10}$$

with:

$$\mathbf{A} = \partial f / \partial p_R |_{p_R = p_0, \mathbf{\Theta} = 0}, \quad \mathbf{F} = \partial f / \partial \mathbf{\Theta} |_{p_R = p_0, \mathbf{\Theta} = 0}.$$

$\mathbf{F}$ can be computed as described in refs. [5,6]. Using eq. (10), eqs. (9) can be rewritten:

$$\mathcal{M} = (f_0 + \mathbf{A} \cdot \Delta p + \mathbf{F} \cdot \mathbf{\Theta} - m)^T \mathbf{V}^{-1} (f_0 + \mathbf{A} \cdot \Delta p + \mathbf{F} \cdot \mathbf{\Theta} - m) + \mathbf{\Theta}^T \mathbf{S}^{-1} \mathbf{\Theta},\tag{11a}$$

with $f_0 = f(p_0, 0)$ and $\Delta p = p_R - p_G$.

Differentiation of $\mathcal{M}$ with respect to $\mathbf{\Theta}$ gives:

$$\mathbf{F}^T \mathbf{V}^{-1} (f_0 + \mathbf{A} \cdot \Delta p - m) + (\mathbf{F}^T \mathbf{V}^{-1} \mathbf{F} + \mathbf{S}^{-1}) \mathbf{\Theta} = 0,$$

or, setting $\mathbf{X} = \mathbf{F}^T \mathbf{V}^{-1} \mathbf{F} + \mathbf{S}^{-1}$:

$$\mathbf{\Theta} = -\mathbf{X}^{-1} \mathbf{F}^T \mathbf{V}^{-1} (f_0 + \mathbf{A} \cdot \Delta p - m).\tag{11b}$$

Substitution of eq. (11b) into eq. (11a) leads to:

$$\mathcal{M} = \left(f_0 + \mathbf{A} \cdot \Delta p - m - \mathbf{FX}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}(f_0 + \mathbf{A} \cdot \Delta p - m) \right)^{\mathsf{T}} \cdot \mathbf{V}^{-1}$$
$$\cdot \left(f_0 + \mathbf{A} \cdot \Delta p - m - \mathbf{FX}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}(f_0 + \mathbf{A} \cdot \Delta p - m) \right)$$
$$+ (f_0 + \mathbf{A} \cdot \Delta p - m)^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{FX}^{-1}\mathbf{S}^{-1}\mathbf{X}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}(f_0 + \mathbf{A} \cdot \Delta p - m),$$

or

$$\mathcal{M} = (f_0 + \mathbf{A} \cdot \Delta p - m)^{\mathsf{T}} \cdot \mathbf{W} \cdot (f_0 + \mathbf{A} \cdot \Delta p - m), \tag{12}$$

with

$$\mathbf{W} = (\mathbf{I} - \mathbf{FX}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1})^{\mathsf{T}}\mathbf{V}^{-1}(\mathbf{I} - \mathbf{FX}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}) + \mathbf{V}^{-1}\mathbf{FX}^{-1}\mathbf{S}^{-1}\mathbf{X}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1},$$

and $\mathbf{I}$ the identity matrix.

The weight matrix $\mathbf{W}$ may be reduced further:

$$\mathbf{W} = \mathbf{V}^{-1} - 2\mathbf{V}^{-1}\mathbf{FX}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1} + \mathbf{V}^{-1}\mathbf{FX}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{FX}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1} + \mathbf{V}^{-1}\mathbf{FX}^{-1}\mathbf{S}^{-1}\mathbf{X}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}$$
$$= \mathbf{V}^{-1} - 2\mathbf{V}^{-1}\mathbf{FX}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1} + \mathbf{V}^{-1}\mathbf{FX}^{-1}(\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{F} + \mathbf{S}^{-1})\mathbf{X}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}$$
$$= \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{FX}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}, \tag{13}$$

with

$$\mathbf{X} = \mathbf{S}^{-1} + \mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{F}.$$

Eq. (12) is just the χ^2 ansatz obtained by the global method (5a), since the covariance matrix computed in eqs. (6) and (7b) can be expressed as $\mathbf{V} + \mathbf{FSF}^{\mathsf{T}}$ in the notation of this section. In fact, $\mathbf{W}$ is the inverse of $\mathbf{V} + \mathbf{FSF}^{\mathsf{T}}$:

$$(\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{FX}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}) \cdot (\mathbf{V} + \mathbf{FSF}^{\mathsf{T}}) = \mathbf{I} - \mathbf{V}^{-1}\mathbf{FX}^{-1}\mathbf{F}^{\mathsf{T}} + \mathbf{V}^{-1}\mathbf{FSF}^{\mathsf{T}} - \mathbf{V}^{-1}\mathbf{FX}^{-1}\mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{FSF}^{\mathsf{T}}$$
$$= \mathbf{I} + \mathbf{V}^{-1}\mathbf{FSF}^{\mathsf{T}} - \mathbf{V}^{-1}\mathbf{FX}^{-1}(\mathbf{I} + \mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{FS})\mathbf{F}^{\mathsf{T}} = \mathbf{I} + \mathbf{V}^{-1}\mathbf{FSF}^{\mathsf{T}} - \mathbf{V}^{-1}\mathbf{FX}^{-1}(\mathbf{S}^{-1} + \mathbf{F}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{F})\mathbf{SF}^{\mathsf{T}} = \mathbf{I}.$$

q.e.d.

It follows that the global method and the break point method are equivalent as far as the estimate of the initial parameter p_R is concerned.

The amount of computation, however, is different for the two methods. If there are n coordinates and m break points, the calculation of $\mathbf{W}$ according to eq. (13) needs about $4m^3 + 8m^2n + 2n^2m$ operations (1 operation = 1 multiplication + 1 addition), whereas the calculation of $\mathbf{W}$ as the inverse of $\mathbf{V} + \mathbf{FSF}^{\mathsf{T}}$ needs about $n^3/2 + 2n^2m$ operations. Therefore, the global method is more efficient than the break point method as soon as there are more than about $n/4$ break points. It does not give, however, the close approximation of the particle's path everywhere along the track that can be achieved by the break point method. As a consequence, the break point method also allows a better judgement of outliers, and is especially adequate for track models like spline-interpolation [8].

A disadvantage of the break point method is the fact that continuous scatterers have to be approximated by layers of material. This might drastically increase the number of break points in a dense material.

2.4. Track element merging by weighted means

In a complex detector it is often necessary that track segments are first fitted separately. The problem of combining the information is discussed in this section. It shall be assumed that there are two detector modules with two estimates $\tilde{p}_i$ of the track parameters at the respective reference planes $z = z_{R_i}$, $i = 1, 2$ (see fig. 5). The covariance matrices of the $\tilde{p}_i$ are designated by $\mathbf{C}_i$ ($i = 1, 2$). The reference plane of the combined track information is assumed to be equal to $z = z_{R_2}$.
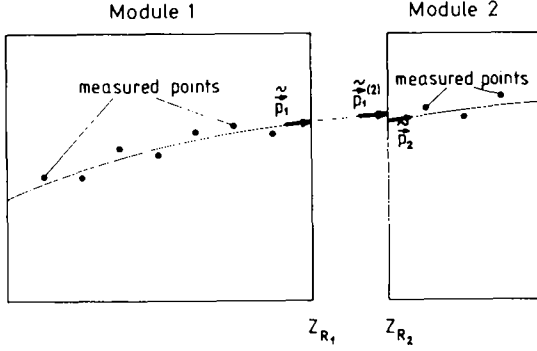
Fig. 5. In track element merging, the track parameters $\tilde{p}_1$ are propagated to a new reference plane. $\tilde{p}_1^{(2)}$ and $\tilde{p}_2$ are considered as direct measurements of the true track parameters at the reference plane.

The idea is now to propagate the estimate $\tilde{p}_1$ and its covariance matrix $\mathbf{C}_1$ to $z = z_{R_2}$:

$$\tilde{p}_1^{(2)} = \tilde{p}_2(\, p_1)$$

$$\mathbf{C}_1' = \mathbf{D} \cdot \mathbf{C}_1 \cdot \mathbf{D}^{\mathrm{T}} \quad \text{with} \quad \mathbf{D} = \partial p_2 / \partial p_1. \tag{14a}$$

Contributions of multiple scattering between z_{R_1} and z_{R_2} (including material at z_{R_1}) are added to $\mathbf{C}_1'$, to give the final covariance matrix $\mathbf{C}_1^{(2)}$ of $\tilde{p}_1^{(2)}$:

$$\mathbf{C}_1^{(2)} = \mathbf{C}_1' + \sum_i \frac{\partial p_2}{\partial(\, p_{1,3},\, p_{1,4})} \frac{\partial(\, p_{1,3},\, p_{1,4})}{\partial(\Theta_{i,1},\, \Theta_{i,2})} \cdot \mathbf{C}(\boldsymbol{\Theta}_i) \frac{\partial(\, p_{1,3},\, p_{1,4})^{\mathrm{T}}}{\partial(\Theta_{i,1},\, \Theta_{i,2})} \left(\frac{\partial p_2}{\partial(\, p_{1,3},\, p_{1,4})} \right)^{\mathrm{T}}$$

$$+ \text{contributions of continuous scattering (see eq. 7b)} \tag{14b}$$

with $\mathbf{C}(\boldsymbol{\Theta}_i)$ the covariance matrix of the scattering angles $\Theta_{i,1}$, $\Theta_{i,2}$, and i the index for all scatterers in the interval $[z_{R_1},\, z_{R_2})$.

If one considers now $\tilde{p}_1^{(2)}$ and $\tilde{p}_2$ as direct measurements of p_2, one can make a joint least-squares ansatz:

$$\mathscr{M}(\, p_2) = (\, p_2 - \tilde{p}_2)^{\mathrm{T}} \mathbf{C}_2^{-1}(\, p_2 - \tilde{p}_2) + (\, p_2 - \tilde{p}_1^{(2)})^{\mathrm{T}} (\mathbf{C}_1^{(2)})^{-1} (\, p_2 - \tilde{p}_1^{(2)}). \tag{15a}$$

The final estimate is a weighted mean:

$$\tilde{p}_2 = \left((\mathbf{C}_1^{(2)})^{-1} + \mathbf{C}_2^{-1}\right)^{-1} \left((\mathbf{C}_1^{(2)})^{-1} \tilde{p}_1^{(2)} + \mathbf{C}_2^{-1} \tilde{p}_2\right), \tag{15b}$$

and $\mathscr{M}(\, p_2)$ is χ^2 distributed with 5 degrees of freedom.

The ansatz (15a) contains an implicit assumption, namely that $\tilde{p}_1^{(2)}$ and $\tilde{p}_2$ are independent. This is, however, only true if the reference plane z_{R_2} is at the near end of module 2, as regarded from module 1. In this case the combined estimate is between the two modules and therefore not of great interest, except for a χ^2 test. In the more interesting case of z_{R_2} being at the far end of module 2, one notes first that the difference

$$p_2(\, p_1^t) - p_2^t,$$

(where t denotes the true values of the track parameters) is a random quantity. It is correlated to the contribution of multiple scattering to the measurement errors in module 2 and hence also to

$$\tilde{p}_2 - p_2^t.$$

Therefore, also $(\tilde{p}_1^{(2)} - p_2^t)$ and $(\tilde{p}_2 - p_2^t)$ are correlated and their covariance matrix has the form:

$$\mathscr{E}\left\{ (\tilde{p}_2 - p_2^t) \cdot (\tilde{p}_1^{(2)} - p_2^t)^{\mathrm{T}} \right\} = -\left(\mathbf{A}_2^{\mathrm{T}} \mathbf{V}_2^{-1} \mathbf{A}_2\right)^{-1} \mathbf{A}_2^{\mathrm{T}} \mathbf{V}_2^{-1} \cdot \mathscr{E}\left\{ \boldsymbol{\epsilon}_2^{ms} \cdot (\, p_2(\, p_1^t) - p_2^t)^{\mathrm{T}} \right\}, \tag{16}$$

where ϵ_2^{ms} is the contribution of multiple scattering in module 2 and $\mathbf{A}_2$, $\mathbf{V}_2$ are the matrices of formula (5b) for module 2.

The evaluation of the expectation in eq. (16) can be quite lengthy, although a certain amount of it can be done during the individual fit in module 2. The problem disappears if there is no multiple scattering in module 2.

2.5. The progressive method

An elegant way out of the dilemma described above is the progressive method of track following proposed in refs. [12,13]. It can be regarded as a special case of track element merging, where the second element consists only of a measurement of two independent coordinates at $z = z_{R_2}$, e.g. m_{2n+1}, m_{2n+2} (n pairs of coordinates are assumed to have been used for track segment 1). In this case the second weight matrix has the simple form:

$$\mathbf{C}_2^{-1} = \begin{pmatrix} 1/\sigma_{2n+1}^2 & 0 & 0 & 0 & 0 \\ 0 & 1/\sigma_{2n+2}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

and

$$\tilde{p}_2 = (m_{2n+1}, m_{2n+2}, 0, 0, 0)^{\mathsf{T}}.$$

Since $\tilde{p}_2 - p_2^1$ is just the intrinsic measurement error of the detector, there is no correlation between $p_1^{(2)}$ and $\tilde{p}_2$, and formula (15b) can be applied. This procedure can be continued iteratively by incorporating measurement after measurement (fig. 6). It should be noted that only the current last estimate of p contains all the information used so far, since the preceding estimates are not updated. Therefore, this method should be used in such a way that one starts from the outer detectors and proceeds towards the vertex region, until finally all the available information has been used.

The method is not restricted to measurements of coordinates but can also be applied to measurements of quantities related only to the momentum (e.g. E, β or m). In this case, only the corresponding element $(\mathbf{C}_2)_{55}$ is not equal to 0.
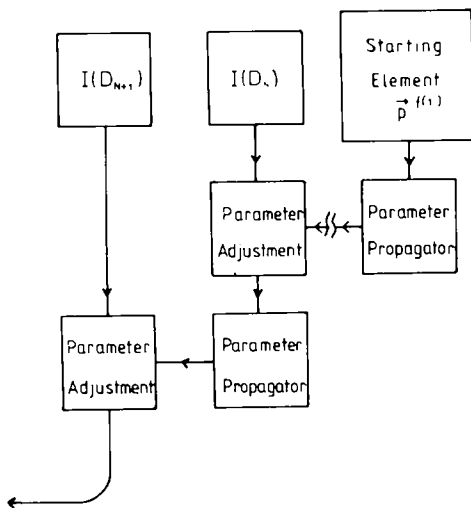

Fig. 6. The progressive method. Starting from the outer detectors, the track is followed step by step towards the center. The estimate of the actual running track parameter is optimised.

During track following it is also of interest to define a cumulative χ^2:

$$\chi^2_{2n+2} = \chi^2_{2n} + \chi^2_{(+)},$$ (17)

where χ^2_{2n} is the χ^2 resulting from the fit of track segment 1 and has an average value of $2n - 5$, while $\chi^2_{(+)}$ is obtained by minimising eq. (15a) and has in this case two degrees of freedom. The average value of χ^2_{2n+2} is therefore $2n - 3$. If the errors are Gaussian, χ^2_{2n} and $\chi^2_{(+)}$ are independent and really χ^2-distributed, and so is then χ^2_{2n+2}.

$\chi^2_{(+)}$ can be used as a fast check whether a point belongs to a track; the cumulative χ^2 gives a judgement of the overall quality of the track. It might be biased towards smaller values, however, if $\chi^2_{(+)}$ is used as a selection criterion.

The main advantage of this method is its efficiency in the case of a large number of coordinates and non-negligible multiple scattering:

– The inversion of a matrix of the dimension of the number of measurements is avoided. Only matrices of dimension 5×5 have to be inverted.
– The decision on outliers can be taken as one goes along the track.
– The real track is followed.
– The linear approximation of the track model needs only to be valid over a short range.

Some drawbacks are:

– The method is "per se" a mixed method, as an initial track segment has to be fitted by some other method. (Two points, however, are enough if the momentum is given weight 0).
– To start track following at the outside requires some care to overcome the following problem: Usually no dense coordinate strings are available in the outer detectors, and the track must be propagated over large interdetector gaps, through thick scatterers, etc.
– The decision on outliers relies only on the information incorporated so far, which might be rather poor at the beginning.
– The asymptotic properties must be checked empirically.

3. Vertex evaluation

3.1. Concepts

A vertex fit serves two purposes. The first is to estimate the position of the point of interaction and the momentum vectors of the tracks emerging from this point (with improved precision due to the vertex constraint). The second is to check the association of tracks to a vertex, i.e. the decision whether the track does indeed originate at this vertex. The following discussion is applicable both to the primary interaction vertex and to an eventual secondary vertex (a decay or secondary interaction). While in the first case the exact position of the vertex might seem a simple mathematical constraint, it is of some importance in the second case, since it determines the direction of the (possibly unseen) track connecting the two vertices.

In both cases the momentum vectors of all emerging tracks should be computed with the best possible precision together with their covariance matrix, since they are the input to a subsequent kinematical fit.

The method described below has been proposed and used successfully by the first generation experiments at the CERN intersecting storage rings [4]. Unfortunately, it involves the inversion of a matrix of the order $3n$ (n = number of tracks). Since the number of arithmetical operations for the inversion increases with the third power of the order, this method becomes prohibitive with a further increase in energy with the consequence of higher multiplicities. Also, the events become more complex with increasing need to eliminate tracks which do not belong to the primary interaction vertex.

Therefore, a new algorithm for the computation of the estimates, their covariances and the χ^2 was developed, allowing to apply this method also to very complex events. In order to avoid unnecessary

repetitions of the vertex fit, a recursive method is also proposed, which allows to check the association of tracks to a common vertex.

The input to the vertex fit consists of information about the tracks to be grouped together. Normally one considers the estimated track parameters at a reference surface (see sect. 2) as virtual measurements. The reference surface will in most cases be either a plane, a cylinder (especially in storage ring experiments) or the beam tube.

The choice of the reference surface has a certain influence on the behaviour of the fit, since it is desirable that the virtual measurements, namely the track parameters, are to a good approximation linear functions of the vertex position and of the parameters determining the momentum vector at the vertex.

In some cases the reference surface will coincide with a physical surface, e.g. the wall of a vacuum vessel or a vertex chamber. If multiple scattering in this wall is important, it can be taken into account easily, by augmenting the covariance matrix of the estimated track parameters (see eq. (14b)).

If the position of the vertex is known to some precision a priori, as is the case for the interaction region of a storage ring, this knowledge can be considered as an independent measurement of the position, with its proper error matrix.

If a single track is poorly defined, its coordinates should be incorporated directly into the vertex fit, instead of a possibly biased estimate of the track parameters. This should also be done if some a priori knowledge of the vertex position was used in a first individual track fit.

3.2. A fast global method

For the vertex fit, the fitted track parameters $\tilde{p}_i$ are regarded as virtual measurements and denoted without $\tilde{\ }$ in the following.

Under the assumption that there are n tracks with the estimated parameters $p_1, \ldots, p_n$ and the corresponding weight matrices $G_1 = C_1^{-1}, \ldots, G_n = C_n^{-1}$, there are $5n$ virtual measurements. The parameters are the vertex position x and n 3-vectors q_i, where q_i is a generalized momentum vector at the vertex (see fig. 7). Therefore, there are $3(n + 1)$ parameters. The charge of a track is assumed to be known from the individual track fit. The functional dependence of $p_1, \ldots, p_n$ on the parameters is now expressed by the
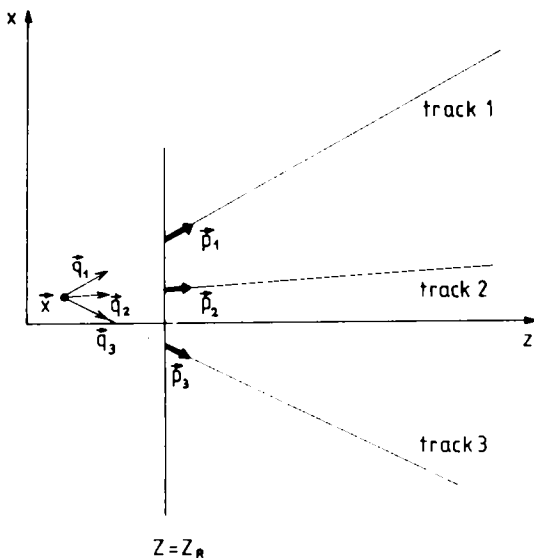

Fig. 7. On the right hand side, the track fitting for single tracks is demonstrated. For the vertex fit, the parameters p_i are considered as "virtual measurements" (left hand side).

following linear model:

$$
\begin{pmatrix} p_1 \\ \cdots \\ p_n \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1 & 0 & & \cdots & 0 \\ \mathbf{A}_2 & 0 & \mathbf{B}_2 & 0 & \cdots & 0 \\ \cdots & \cdots & 0 & & \cdots & 0 \\ & \cdots & \cdots & & & \cdots \\ \mathbf{A}_n & 0 & 0 & 0 & \cdots & \mathbf{B}_n \end{pmatrix} \cdot \begin{pmatrix} x \\ q_1 \\ \cdots \\ q_n \end{pmatrix} + \begin{pmatrix} c_1 \\ \cdots \\ c_n \end{pmatrix},
\tag{18}
$$

with:

$$
\mathbf{A}_i = \partial p_i / \partial x \big|_{x_0, q_{i0}}, \quad \mathbf{B}_i = \partial p_i / \partial q_i \big|_{x_0, q_{i0}},
$$

x_0, q_{i0} approximations of x and q_i (expansion point); $c_i = p_i(x_0, q_{i0}) - \mathbf{A}_i x_0 - \mathbf{B}_i q_{i0}$, constants which are omitted without loss of generality: in the following, p_i stands for $p_i - c_i$.

$\mathbf{A}_i$ and $\mathbf{B}_i$ are 5×3 matrices. $\mathbf{A}_i$ depends also on q_i and $\mathbf{B}_i$ depends also on x. It is therefore mandatory that a good approximation of all parameters is available. The usual least-squares ansatz

$$
\mathcal{M}(x, q_1, \ldots, q_n) = \sum_{j=1}^{n} \left(p_j - \mathbf{A}_j x - \mathbf{B}_j q_j \right)^{\mathsf{T}} \mathbf{G}_j \left(p_j - \mathbf{A}_j x - \mathbf{B}_j q_j \right)
\tag{19}
$$

yields the following estimates of the parameters:

$$
\begin{pmatrix} \tilde{x} \\ \tilde{q}_1 \\ \cdots \\ \tilde{q}_n \end{pmatrix} = \mathbf{M}^{-1} \cdot \mathbf{N} \cdot \begin{pmatrix} p_1 \\ \cdots \\ p_n \end{pmatrix},
\tag{20a}
$$

with:

$$
\mathbf{M} = \begin{pmatrix} \mathbf{D}_0 & \mathbf{D}_1 & \cdots & \mathbf{D}_n \\ \mathbf{D}_1^{\mathsf{T}} & \mathbf{E}_1 & & \\ \cdots & & 0 & \\ & & 0 & \\ \mathbf{D}_n^{\mathsf{T}} & & & \mathbf{E}_n \end{pmatrix},
\tag{20b}
$$

$$
\mathbf{D}_0 = \sum_{i=1}^{n} \mathbf{A}_i^{\mathsf{T}} \mathbf{G}_i \mathbf{A}_i,
\tag{20c}
$$

$$
\mathbf{D}_i = \mathbf{A}_i^{\mathsf{T}} \mathbf{G}_i \mathbf{B}_i, \qquad \mathbf{E}_i = \mathbf{B}_i^{\mathsf{T}} \mathbf{G}_i \mathbf{B}_i, \quad i > 0,
$$

and with:

$$
\mathbf{N} = \begin{pmatrix} \mathbf{A}_1^{\mathsf{T}} \mathbf{G}_1 & \mathbf{A}_2^{\mathsf{T}} \mathbf{G}_2 & \cdots & \mathbf{A}_n^{\mathsf{T}} \mathbf{G}_n \\ \mathbf{B}_1^{\mathsf{T}} \mathbf{G}_1 & 0 & & 0 \\ 0 & \cdots & 0 & \\ & 0 & \cdots & 0 \\ 0 & & 0 & \mathbf{B}_n^{\mathsf{T}} \mathbf{G}_n \end{pmatrix}.
\tag{20d}
$$

If we write $\mathbf{M}^{-1}$ in the form:

$$
\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{C}_{00} & \cdots & \mathbf{C}_{0n} \\ \cdots & & \cdots \\ \mathbf{C}_{n0} & \cdots & \mathbf{C}_{nn} \end{pmatrix},
\tag{21a}
$$

the submatrices C_{ij} can be computed explicitly:

$$C_{(x)} = \left(D_0 - \sum_{i=1}^{n} D_i E_i^{-1} D_i^{T} \right)^{-1}.$$

$$C_{0j} = -C_{(x)} D_j E_j^{-1}, \qquad C_{j0} = C_{0j}^{T}, \qquad j > 0,$$ (21b)

$$C_{ij} = \delta_{ij} E_i^{-1} + E_i^{-1} D_i^{T} C_{(x)} D_j E_j^{-1} = \delta_{ij} E_i^{-1} - E_i^{-1} D_i^{T} C_{0j}, \qquad i, j > 0.$$

From eq. (20a) it follows that M^{-1} is the covariance matrix of the fitted parameters:

$$C_{(x)} = \text{cov}(\tilde{x}, \tilde{x}); \qquad C_{0j} = \text{cov}(\tilde{x}, \tilde{q}_j), \qquad j > 0,$$ (21c)

$$C_{ij} = \text{cov}(\tilde{q}_i, \tilde{q}_j), \qquad i, j > 0.$$

It is important to note that the fitted parameters $\tilde{x}$ and $\tilde{q}_i$ can be calculated without evaluation of the full covariance matrix M^{-1}.

Substitution of eq. (21b) into eq. (20a) yields the following expressions for $\tilde{x}$ and $\tilde{q}_i$:

$$x = C_{(x)} \cdot \sum_{j=1}^{n} A_j^{T} G_j \left(I - B_j E_j^{-1} B_j^{T} G_j \right) p_j,$$ (22a)

$$\tilde{q}_i = -E_i^{-1} D_i^{T} \tilde{x} + E_i^{-1} B_i^{T} G_i p_i = E_i^{-1} B_i^{T} G_i (-A_i \tilde{x} + p_i).$$ (22b)

The amount of computation needed is proportional to the number of tracks and is in the order of a few hundred times n operations (as defined in section 2.3).

The χ^2 of the fit is given by substituting eq. (20a) into eq. (19):

$$\chi^2 = \sum_{j=1}^{n} (p_j - \tilde{p}_j)^{T} G_j (p_j - \tilde{p}_j),$$ (22c)

with

$$\tilde{p}_j = A_j \tilde{x} + B_j \tilde{q}_j.$$ (22d)

It can be computed with less than $100n$ additional operations. It is essential to perform the χ^2 test before the calculation of the covariance matrix which needs another $(27/2 \, n^2 + 40n)$ operations. Note that a standard inversion of M alone would need $\frac{1}{2}(3n + 3)^3 \approx 27/2 n^3 + 40n^2$ operations! Hence this method yields a substantial gain already for vertices with 3 or 4 tracks, and can be applied equally well to V_0's.

If there is an independent measurement v of the vertex position (e.g. knowledge of the beam profile obtained from elastic scattering data) with the weight matrix G, eq. (22a) has to be modified:

$$\tilde{x} = C_{(x)} \left[Gv + \sum_{j=1}^{n} A_j^{T} G_j \left(I - B_j E_j^{-1} B_j^{T} G_j \right) p_j \right].$$ (23)

Eq. (22b) holds without change. In the computation of $C_{(x)}$ the matrix D_0 has to be replaced by $D_0 + G$.

3.3. Track association

Although the method presented in sect. 3.2 is much faster than a conventional least-squares fit, it is still not economical to find the association of tracks to a common vertex by repeated application of the global fit to different subsets of tracks.

Instead, we propose an iterative procedure of track association, in analogy to the progressive method of sect. 2.4:

Let us assume that a vertex has been reconstructed with m tracks, giving an estimate $\tilde{x}^{(m)}$ and its covariance matrix $C_{(x)}^{(m)}$. One needs now to check whether track $m + 1$ is compatible with this vertex. To

this end, we consider $\tilde{x}^{(m)}$ and p_{m+1} as virtual measurements and fit the parameters x and q_{m+1} by minimizing the χ^2:

$$\chi^2_{(+)} = \left(p_{m+1} - \mathbf{A}_{m+1}x - \mathbf{B}_{m+1}q_{m+1} \right)^{\mathsf{T}} \mathbf{G}_{m+1} \left(p_{m+1} - \mathbf{A}_{m+1}x - \mathbf{B}_{m+1}q_{m+1} \right)$$

$$+ \left(\tilde{x}^{(m)} - x \right)^{\mathsf{T}} \left(\mathbf{C}_{00}^{(m)} \right)^{-1} \left(\tilde{x}^{(m)} - x \right). \tag{24}$$

It turns out that the solution is given just by the formulae (22a), (22b), which yield:

$$\tilde{x}^{(m+1)} = \mathbf{C}_{00}^{(m+1)} \left[\left(\mathbf{C}_{00}^{(m)} \right)^{-1} \tilde{x}^{(m)} + \mathbf{A}_{m+1}^{\mathsf{T}} \mathbf{G}_{m+1} \left(\mathbf{I} - \mathbf{B}_{m+1} \mathbf{E}_{m+1}^{-1} \mathbf{B}_{m+1}^{\mathsf{T}} \mathbf{G}_{m+1} \right) p_{m+1} \right], \tag{25a}$$

$$\tilde{q}_{m+1} = \mathbf{E}_{m+1}^{-1} \mathbf{B}_{m+1}^{\mathsf{T}} \mathbf{G}_{m+1} \left(-\mathbf{A}_{m+1} \tilde{x}^{(m+1)} + p_{m+1} \right), \tag{25b}$$

using (see eqs. (20c) and (21b)):

$$\mathbf{C}_{00}^{(m+1)} = \left[\left(\mathbf{C}_{00}^{(m)} \right)^{-1} + \mathbf{A}_{m+1}^{\mathsf{T}} \mathbf{G}_{m+1} \mathbf{A}_{m+1} - \mathbf{D}_{m+1} \mathbf{E}_{m+1}^{-1} \mathbf{D}_{m+1}^{\mathsf{T}} \right]^{-1} \tag{25c}$$

$$= \left[\left(\mathbf{C}_{00}^{(m)} \right)^{-1} + \mathbf{A}_{m+1}^{\mathsf{T}} \mathbf{G}_{m+1} \left(\mathbf{I} - \mathbf{B}_{m+1} \mathbf{E}_{m+1}^{-1} \mathbf{B}_{m+1}^{\mathsf{T}} \mathbf{G}_{m+1} \right) \mathbf{A}_{m+1} \right]^{-1}.$$

Therefore, this fit is equivalent to eqs. (22a,b) and hence to a complete global fit with $m+1$ tracks. The global χ^2 can be updated:

$$\chi^2_{m+1} = \chi^2_m + \chi^2_{(+)}. \tag{26a}$$

A proof of this fact is sketched at the end of this section. Note that $\chi^2_{(+)}$ can be computed using only the fitted values $\tilde{x}^{(m+1)}$ and $\tilde{q}_{m+1}$.

It is normally not necessary to update the remaining estimated momentum vectors $q_1^{(m)}, \dots, q_m^{(m)}$, since it is more economical to recompute all $\tilde{q}_i$ by means of eq. (22b) at the end. It can easily be done, however, by using eq. (22b):

$$\tilde{q}_i^{(m+1)} = \tilde{q}_i^{(m)} - \mathbf{E}_i^{-1} \mathbf{D}_i^{\mathsf{T}} \left(\tilde{x}^{(m+1)} - \tilde{x}^{(m)} \right). \tag{26b}$$

$\chi^2_{(+)}$ is χ^2-distributed with two degrees of freedom and can be used as a test criterion for the decision whether track $m+1$ is compatible with the vertex $\tilde{x}^{(m)}$. By the same algorithm a track can also be removed from a vertex by changing the sign of its weight matrix $\mathbf{G}_i$.

Secondary vertices – if properly recognized by the pattern recognition – bear no new features and can be treated exactly in the same way as the primary vertex.

A proof of eq. (26a) can be obtained in the following way. $\chi^2_{(+)}$ can be written as:

$$\chi^2_{(+)} = \left(p_{m+1} - \tilde{p}_{m+1} \right)^{\mathsf{T}} \mathbf{G}_{m+1} \left(p_{m+1} - \tilde{p}_{m+1} \right) + \left(\tilde{x}^{(m)} - \tilde{x}^{(m+1)} \right)^{\mathsf{T}} \left(\mathbf{C}_{00}^{(m)} \right)^{-1} \left(\tilde{x}^{(m)} - \tilde{x}^{(m+1)} \right). \tag{27a}$$

We show that the difference $\chi^2_{m+1} - \chi^2_m$ is equal to eq. (27a):

$$\chi^2_{m+1} - \chi^2_m = \sum_{i=1}^{m+1} \left(p_i - \tilde{p}_i^{(m+1)} \right)^{\mathsf{T}} \mathbf{G}_i \left(p_i - \tilde{p}_i^{(m+1)} \right) - \sum_{i=1}^{m} \left(p_i - \tilde{p}_i^{(m)} \right)^{\mathsf{T}} \mathbf{G}_i \left(p_i - \tilde{p}_i^{(m)} \right)$$

$$= \left(\tilde{p}_{m+1} - \tilde{p}_{m+1}^{(m+1)} \right)^{\mathsf{T}} \mathbf{G}_{m+1} \left(p_{m+1} - \tilde{p}_{m+1}^{(m+1)} \right) + 2 \sum_{i=1}^{m} \left(\tilde{p}_i^{(m)} - \tilde{p}_i^{(m+1)} \right)^{\mathsf{T}} \mathbf{G}_i \left(p_i - \tilde{p}_i^{(m)} \right)$$

$$+ \sum_{i=1}^{m} \left(\tilde{p}_i^{(m)} - \tilde{p}_i^{(m+1)} \right)^{\mathsf{T}} \mathbf{G}_i \left(\tilde{p}_i^{(m)} - \tilde{p}_i^{(m+1)} \right). \tag{27b}$$

The first terms in eqs. (27a) and (27b) are equal. By using eqs. (22b) and (22d) it follows that

$$\tilde{p}_i^{(m)} = \mathbf{A}_i \tilde{x}^{(m)} + \mathbf{B}_i \tilde{q}_i^{(m)} = \left(\mathbf{A}_i - \mathbf{B}_i \mathbf{E}_i^{-1} \mathbf{D}_i^{\mathsf{T}} \right) \tilde{x}^{(m)} + \mathbf{B}_i \mathbf{E}_i^{-1} \mathbf{B}_i^{\mathsf{T}} \mathbf{G}_i p_i, \tag{28a}$$
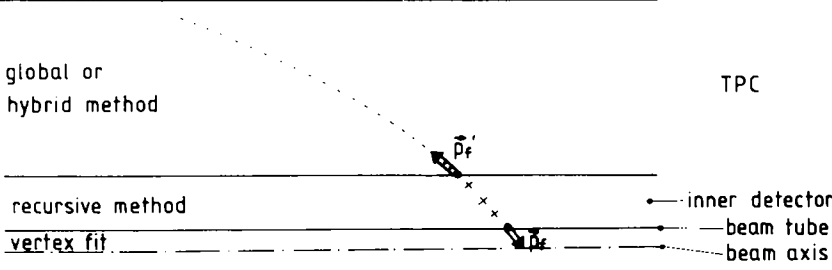
Fig. 8. An example for a program structure is shown. After a global fit in the TPC and a recursive track following through a high precision detector, the "virtual measurements" for the subsequent vertex fit are obtained.

and

$$\Delta \tilde{p}_i = \tilde{p}_i^{(m)} - \tilde{p}_i^{(m+1)} = \left(\mathbf{A}_i - \mathbf{B}_i \mathbf{E}_i^{-1} \mathbf{D}_i^{\mathrm{T}} \right) \left(\tilde{x}^{(m)} - \tilde{x}^{(m+1)} \right). \tag{28b}$$

By substituting eqs. (28a), (28b) and (22a) into eq. (27b), it turns out that the second term is equal to 0, and that the third term is equal to:

$$\left(\tilde{x}^{(m)} - \tilde{x}^{(m+1)} \right)^{\mathrm{T}} \cdot \sum_{i=1}^{m} \left(\mathbf{A}_i^{\mathrm{T}} \mathbf{G}_i \mathbf{A}_i - \mathbf{D}_i \mathbf{E}_i^{-1} \mathbf{D}_i^{\mathrm{T}} \right) \cdot \left(\tilde{x}^{(m)} - \tilde{x}^{(m+1)} \right). \tag{29}$$

A glance at eqs. (20c) and (21b) shows that eq. (29) is equal to the second term in eq. (27a). Hence eqs. (27a) and (27b) are identical; q.e.d.

4. An example

One of the detectors (DELPHI) at the large electron positron collider (LEP) serves as an example for the application of the methods discussed above. In this detector, a time projection chamber (TPC), giving a high number of measured space points for large angle tracks, is surrounded by several outer detector modules giving less powerful measurements. Inside the TPC, a high precision inner detector measuring a small number of coordinates improves the precision towards the vertex region (fig. 8).

The natural choice for obtaining a track segment to start with is the TPC, because of the large number of measurements, the small multiple scattering and the highly homogeneous magnetic field. In this detector the progressive method might serve as a powerful check of the pattern recognition, particularly in the case of high multiplicities, but for the final track fit the global method seems to be the most appropriate one. Information from some of the outer detectors will be merged in by a combination of these methods. This will be discussed somewhere else.

The information provided by the inner detector will be added by the progressive method. This procedure ensures that tracks close to each other in the vicinity of the vertex region can be well separated. Also, the estimate of the final track parameters is optimized where it ought to be, namely at the beam tube.

Acknowledgments

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