MAGNETIC MIRROR

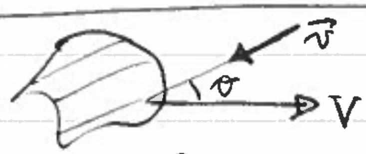
Hp: CONSERVATION OF MAGNETIC MOMENT

If the B field varies adiabatically, then $\frac{p_\perp^2}{B} = \text{cost}$

Since B does no work $p^2 = p_\perp^2 + p_\parallel^2 = \text{cost}$

In a region where B increases, $p_\perp^2 \uparrow \Rightarrow p_\parallel^2 \downarrow$, and the particle may end up being reflected

FERMI ACCELERATION



LAB FRAME: E_i, p_i; $\left[\begin{array}{l} pc = \beta E \\ p_i \simeq \beta E \end{array}\right.$

In the MIRROR FRAME

$$\begin{cases} E_i' = \gamma_V(\beta_V p_{x_i} + E_i) \\ p_{x_i}' = \gamma_V(p_{x_i} + \beta_V E) \end{cases} \xRightarrow{\text{REFLECTION}} \begin{cases} E_f' = E_i' \\ p_{x,f}' = -p_{x,i}' \end{cases}$$

Back in the LAB FRAME:

$$E_f = \gamma_{-V}(\beta_{-V} p_{x,f}' + E_f') \qquad [\beta_{-V} = -\beta_V;$$

$$= \gamma_V(\beta_V p_{x_i}' + E_i') \qquad [p_{x,i} = \beta E \cos\vartheta$$

$$E_f = \gamma_V^2 E_i \left[2\beta_V \beta \cos\vartheta + 1 + \beta_V^2\right]$$

$$\Delta E = E_f - E_i = E_i\left[\gamma_V^2(2\beta_V\beta\cos\vartheta + \beta_V^2) + \underbrace{\gamma_V^2 - 1}_{\beta_V^2\gamma_V^2}\right]$$

$$\frac{\Delta E}{E} = 2\beta\beta_V\cos\vartheta + 2\beta_V^2$$

$$\boxed{\frac{\Delta E}{E} = 2\beta_V(\beta\cos\vartheta + \beta_V)}$$

$$w_{TOT} = \frac{u + v_{rel}}{1 + \frac{u v_{rel}}{c^2}} \qquad v_{rel} = \frac{w_{TOT} - u}{1 - \frac{w_{TOT} u}{c^2}}$$

In general the # of collisions per unit time is proportional to the relative velocity between the particle and the mirror:

$$dP(\cos\vartheta) \propto \frac{V\cos\vartheta + v}{1 + vV\cos\vartheta/c^2} \sin\vartheta \, d\varphi \, d\vartheta \propto \frac{V\cos\vartheta + v}{1 + vV\cos\vartheta/c^2} \, d\cos\vartheta$$

Hp: $v \sim c$ and $v \gg V$ (at the $O(1)$ in $\frac{V}{c}$)

$$P(\cos\vartheta)\, d\cos\vartheta = \frac{1}{2}\left(1 + \frac{V\cos\vartheta}{c}\right) d\cos\vartheta \propto \left[1 + \beta_V \cos\vartheta \right.$$

$x = \cos\vartheta$]

$$\langle \cos\vartheta \rangle_P = \frac{\int_{-1}^{1} x(\cancel{1} + \beta_V x)\, dx}{\int_{-1}^{1} (1 + \beta_V) x \, dx} = \frac{\frac{x^3}{3}\beta_V \Big|_{-1}^{1}}{\frac{x^2}{2}\Big|_{-1}^{1}} = \beta_V \frac{\frac{2}{3}}{2} = \frac{\beta_V}{3}$$

And finally

$$\left\langle \frac{\Delta E}{E} \right\rangle_P = \frac{2\beta_V^2}{3} + 2\beta_V^2 = \frac{8}{3}\beta_V^2$$

$$\boxed{\left\langle \frac{\Delta E}{E} \right\rangle_{II} = \frac{8}{3}\beta_V^2}$$

II order Fermi acceleration

I If we are at a shock, assuming isotropy, $P(\cos\vartheta) \propto \cos\vartheta$, which implies:

$$\langle \cos\vartheta \rangle_P = \frac{\int_0^1 x \cdot x \, dx}{\int_0^1 x \, dx} = \frac{\frac{x^3}{3}\Big|_0^1}{\frac{x^2}{2}\Big|_0^1} = \frac{2}{3}, \quad \text{so that}$$

$$\left\langle \frac{\Delta E}{E} \right\rangle_P = \frac{4}{3}\beta_V \beta + 2\beta_V^2 \qquad \text{I order Fermi acceleration}$$

CAVEAT: If $v \gg V$ there may be 2 averages involved [still assuming specular reflection!]

SHOCK HYDRODYNAMICS

Let us start from the conservation eqs for an IDEAL FLUID, described by $\left\{\vec{v}(\vec{x},t);\ \rho(\vec{x},t);\ p(\vec{x},t)\right\}$

MASS: $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$

MOMENTUM: $\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho}\ \left(+\frac{p}{\rho}\,\text{cst}\right) = -\vec{\nabla}w$

ENERGY: $\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2} + \rho E\right) + \vec{\nabla}\cdot\left[\rho\vec{v}\left(\frac{v^2}{2}+w\right)\right] = 0$

$w = \frac{\text{enthalpy}}{\text{unit mass}} = E + pV;\ V = \frac{1}{\rho}$

$w = \frac{\gamma}{\gamma-1}\frac{P}{\rho} = \frac{c_s^2}{\gamma-1}$

$E = \frac{1}{\gamma-1}\frac{P}{\rho} = \frac{c_s^2}{\gamma(\gamma-1)}$

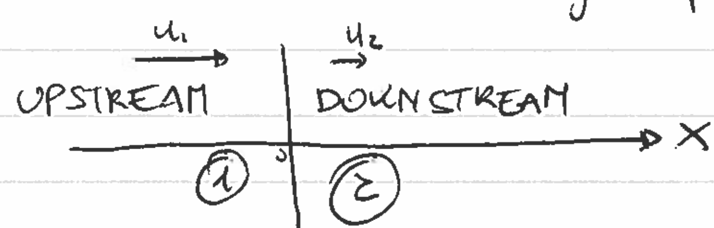
Assume: 1D + stationarity ($\vec{v} \to u$)

MASS: $\rho u = \text{cost}$

MOMENTUM: $\rho u^2 + P = \text{cost}$

ENERGY: $\rho u\left(\frac{u^2}{2}+w\right) = \text{cost} \Rightarrow \frac{1}{2}\rho u^3 + \frac{\gamma}{\gamma-1}uP = \text{cost}$

Let's consider a solution with a jump (and $\Delta S>0$)



RANKINE-HUGONIOT CONDITIONS

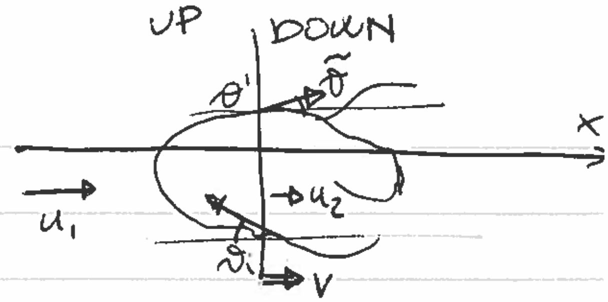
$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\gamma+1}{\gamma-1+2/M_1^2} \qquad M_1^2 = \frac{u_1^2}{c_{s1}^2}$$

$$\frac{P_1}{\rho_1 u_1^2} = \frac{1}{\gamma M_1^2} \qquad P\rho^{-\gamma} = \text{cost}$$

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}; \qquad \frac{T_2}{T_1} = \frac{P_2}{P_1}\cdot\frac{\rho_1}{\rho_2}$$

$M_1^2 \gg 1$:

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} \overset{\gamma=5/3}{=} 4; \qquad \frac{P_2}{P_1} = \frac{2\gamma M_1^2}{\gamma+1} \propto M_1^2; \qquad \frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)M_1^2}{(\gamma+1)^2} \propto M_1^2; \qquad M_2^2 = \frac{\gamma-1}{2\gamma} \overset{\gamma=5/3}{=} \frac{1}{5}$$



SHOCK ACCELERATION

Consider a TEST-PARTICLE
in the DS FRAME, with velocity $\beta \gg u_1, u_2$.
Lp Let us assume that there is TURBULENCE to scatter the part. in pitch angle ϑ.
Let the flight cosine be $\mu = \cos\vartheta$
and be $V = u_1 - u_2$ the relative velocity between UP/DS

$\boxed{DS_i}$ $E = E_i$ $\quad p_{x_i} = \beta \mu_i E_i$ $\qquad pc = \beta E \quad p_{x_i} = p\mu_i$

$\boxed{DS_i \to US}$ $E' = \Gamma E_i (1 + \beta_V \beta \mu_i)$ $\qquad \Gamma = \gamma_V$

The part. is isotropized by turbulence, and may re-encounter the shock with a different $\vartheta = \vartheta'$ and with $\beta = \beta'$

$\hookrightarrow$ $E'_f = E'_i$ $\quad$ & $p'_{xf} = E'_f \beta' \mu'$

Back in the DS.

$\boxed{DS_f}$ $\boxed{E_f = \Gamma^2 E_i (1 + \beta_V \beta \mu_i)(1 - \beta_V \beta' \mu')}$ $\qquad [\beta_{-V} = -\beta_V$

Now, I have to also convert μ' into μ_f

$$\mu_f = -\frac{\beta_V - \mu'\beta}{1 - \mu'\beta_V} \qquad \mu' = \frac{\mu_f + \beta_V}{\beta + \mu_f \beta_V} \qquad [\beta, \beta' \sim 1$$

Finally:
$$\frac{E_f}{E_i} = \Gamma^2 (1 + \beta_V \mu_i)\left[1 - \beta_V \frac{\mu_f + \beta_V}{\beta + \mu_f \beta_V}\right]$$
$$= \cancel{\Gamma^2} (1 + \beta_V \mu_i) \frac{\cancel{1 - \beta_V^2}}{1 + \mu_f \beta_V} = \frac{1 + \beta_V \mu_i}{1 + \beta_V \mu_f}$$

$$\boxed{\frac{E_f}{E_i} \simeq (1 + \beta_V \mu_i)(1 - \beta_V \mu_f)} \quad \text{for } \beta_V \ll 1$$

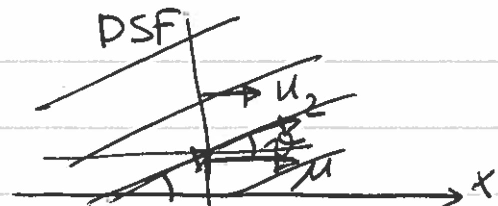
AVERAGE OVER μ_i, μ_f

DS We want to calculate the flux of particle in direction μ : $\phi \propto n v_x \propto \mu$; therefore

$$\left\langle \frac{E_f}{E_i} \right\rangle_{\mu_f} = \frac{\int d\mu_f \mu_f (1+\beta_v \mu_i)(1-\mu_f \beta_v)}{\int d\mu_f \mu_f} \qquad P(\mu) \propto \mu$$

DS Extremes of integration

The rel velocity between the part and the shock is - if the part distribution is isotropic in the DSF - is $\mu_f + u_2/c$



UP→DS ~~DS→UP~~ For the part to enter downstream:

$$\mu_f + u_2 \geq 0 \quad \Rightarrow \quad \boxed{-u_2 \leq \mu_f \leq 1}$$

DS→UP For the part to enter upstream:

$$\mu_i + u_2 \leq 0 \quad \Rightarrow \quad \boxed{-1 \leq \mu_i \leq -u_2}$$

Performing the average:

$\beta_v \equiv V$

$$\left\langle \frac{E_f}{E_i} \right\rangle_{\mu_f} = \frac{1+V\mu_i}{\frac{1}{2}(1-u_2^2)} \int_{-u_2}^{1} d\mu_f (1-\mu_f V)\mu_f \qquad \left[\int_{-u_2}^{1} d\mu\, \mu = \frac{1}{2}(1-u_2^2)\right.$$

$$= \frac{1+V\mu_i}{\frac{1}{2}(1-u_2^2)} \left\{ \frac{1}{2}(1-u_2^2) - \frac{V}{3}(1-u_2^3) \right\} \qquad \left[\int_{-u_2}^{1} d\mu\, \mu^2 = \frac{1+u_2^3}{3}\right.$$

$$\left\langle \frac{E_f}{E_i} \right\rangle_{\mu_f, \mu_i} = \left\{ 1 - \frac{2}{3}V\left(\frac{1+u_2^3}{1-u_2^2}\right) \right\} \frac{1}{\frac{u_2^2-1}{2}} \int_{-1}^{-u_2} d\mu_i\, \mu_i (1+V\mu_i) \qquad \left[\int_{-1}^{-u_2} d\mu\, \mu = \frac{u_2^2-1}{2}\right.$$

$$= \left[1 - \frac{2}{3}V\left(\frac{1+u_2^3}{1-u_2^2}\right)\right]\left[1+\frac{2}{3}V\left(\frac{1-u_2^3}{u_2^2-1}\right)\right] \simeq \qquad \left[\int_{-1}^{-u_2} d\mu\, \mu^2 = \frac{1-u_2^3}{3}\right.$$

$$\simeq 1 + \frac{2}{3}V\left(\frac{1+u_2^3}{u_2^2-1}\right) - \frac{2}{3}V\left(\frac{1-u_2^3}{1-u_2^2}\right) + O(V^2) \simeq 1 + \frac{4}{3}V \qquad \boxed{\left\langle \frac{\Delta E}{E} \right\rangle_{\mu_i, \mu_f} = \frac{4}{3}(u_1 - u_2)}$$

BELL'S APPROACH

Let's start with N_0 particles with energy E_0

Be: – G: the energy gain per step: $E_K = G^K E_0$

– P: the probability of staying in the accelerator

After K cycles: $N_K = P^K N_0$ [G, P indep of E]

Since:

$$\frac{N_K}{N_0} = P^K \quad \log\left(\frac{N_K}{N_0}\right) = K \log P$$
$$\frac{E_K}{E_0} = G^K \quad \log\left(\frac{E_K}{E_0}\right) = K \log G \quad \Rightarrow$$

$$\log\left(\frac{N_K}{N_0}\right) = \frac{\log P}{\log G} \log\left(\frac{E_K}{E_0}\right) \Rightarrow \cancel{\log\left(\frac{N_K}{N_0}\right)} \boxed{\frac{N_K}{N_0} = \left(\frac{E_K}{E_0}\right)^{-\alpha}, \quad \alpha = -\frac{\log P}{\log G}}$$

RETURN PROBABILITY

The ret. prob. P is the ratio of FLUXES of particles returning to the shock to entering part

$$P = \frac{|J_{ret}|}{J_{in}} = \frac{-\int d\mu (\mu + u_2)\big|_{\mu + u_2 \leq 0}}{\int d\mu(\mu + u_2)\big|_{\mu + u_2 \geq 0}} = \frac{-\int_{-1}^{-u_2}}{\int_{-u_2}^{1}} \Leftarrow \left[\int d\mu (\mu + u_2) = \frac{\mu^2}{2} + \mu u_2 \Big|_{min}^{max}\right.$$

$$= \frac{-\left|\mu u_2 + \frac{\mu^2}{2}\right|_{-1}^{-u_2}}{\mu u_2 + \frac{\mu^2}{2}\Big|_{-u_2}^{1}} = \frac{-\left|-u_2^2 + \frac{u_2^2}{2} + u_2 - \frac{1}{2}\right|}{u_2 + \frac{1}{2} + u_2^2 - \frac{u_2^2}{2}} = \frac{-|-u_2^2 + 2u_2 - 1|}{u_2^2 + 2u_2 + 1} = \left(\frac{1 - u_2}{1 + u_2}\right)^2$$

$\boxed{P \simeq 1 - 4u_2}$ for $u_2 \ll 1$ $\quad [r = \frac{u_1}{u_2}$

FIRST ORDER FERMI 2. SHOCKS

$$\alpha = -\frac{\log P}{\log G} = -\frac{\log(1 - 4u_2)}{\log\left(1 + \frac{4}{3}(u_1 - u_2)\right)} \simeq \frac{4u_2}{\frac{4}{3}(u_1 - u_2)} = \frac{3u_2}{u_1 - u_2} = \frac{3}{r - 1}$$

$$\frac{N_K}{N_0} = \left(\frac{E_K}{E_0}\right)^{-\frac{3}{r-1}} \Rightarrow N(>E_0) = N_0\left(\frac{E}{E_0}\right)^{-\frac{3}{r-1}} \Rightarrow \boxed{\frac{dN}{dE} \propto \left(\frac{E}{E_0}\right)^{-\alpha - 1} \propto \left(\frac{E}{E_0}\right)^{-\frac{r+2}{r-1}}}$$