# VISAR Analysis Theory and Mathematical Formalism

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March 30, 2016

## 1 Overview of VISAR Analysis

Line-imaging VISAR (Velocity Interferometer System for Any Reflector) is a useful diagnostic for measuring the shock wave velocity of matter in extreme conditions and is used in many laboratories as a tool to study high pressure, shock, and Z-pinch physics. MEC uses VISAR analysis to study shock waves propagating through materials subject to laser-driven plasma-induced shocks and is capable of measuring shock velocities on a nanosecond timescale.

### 1.1 History

VISAR was developed in the late 1960s to early 1970s as a tool for measuring the velocity history of free surface motion in plate-impact experiments and of projectiles[2]. At that time, VISAR analysis utilized laser interferometry to obtain Velocity vs. Time profiles for any spectrally or diffusely reflecting surface in motion by measuring the resulting shift in fringe pattern produced by Doppler shifted light due to the motion. However, the only way to actually make a meaurement was via photomultiplier tubes and oscilloscopes. A decade later, in the early 1980s, an optical version of VISAR was developed[4], which took advantage of high-speed electronic streak cameras to record interferometer fringe motion. This was named ORVIS, for Optically Recording Velocity Interferometer System. This led to the development of line-imaging VISAR[7], which utilizes the ability of electronic streak cameras to resolve both the spatial and temporal components of the resulting fringe pattern, thus allowing for spatial information to be recorded as well.

### 1.2 Overview

The basic concept of VISAR analysis involves using the Doppler shift of light being reflected off of the surface of a moving object to measure the velocity profile of the object. There are two fundamental components to this analysis: 1) determining the fringe shift of light reflected off of the object, and 2) calculating the velocity of the object based on the measured fringe shift. Further analysis includes calculating the internal pressures exerted on the object based on its velocity profile and an equation of state for that material. However, this calculation goes beyond the scope of basic VISAR analysis and will not be discussed until later.

In order to measure the fringe shift of a moving object, a laser emitting coherent light is used to illuminate the surface of the object in motion. The reflected light is then passed through an interferometer, with one arm of the interferometer being temporally delay with respect to the other arm, creating a fringe pattern based on the relative phases of the two arms; this spatially resolved fringe pattern is measured by a streak camera as a function of time. A shift in the fringe pattern correlates to the motion of the object, therefore the object's velocity is directly proportional to the fringe shift recorded by the camera. So, to determine the velocity profile of the object, the magnitude of the fringe shift must be calculated. The object in motion can be a shock front, a free surface, or a surface behind a transparent window.

# 2 Theory behind VISAR

## 2.1 Doppler Effect

VISAR analysis is fundamentally built upon the principles of the relativistic Doppler effect. For a coherent laser of wavelength $\lambda_0$ incident upon a moving object, the wavelength $\lambda$ of the reflected light is given by:

$$\frac{\lambda}{\lambda_0} = \sqrt{\frac{1 - v/c}{1 + v/c}},$$

where $v$ is the velocity of the object relative to the observer and $c$ is the speed of light. Since it is usually the case that $v << c$, the above formula approximates to the following*:

$$\frac{\lambda}{\lambda_0} \approx 1 - v/c.$$

Therefore, it follows from the above derivation that the velocity of the object in motion is proportionaly to its Doppler shift in wavelength. This Doppler shifted wavelength is measured as a fringe shift by the streak camera after passing through the interferometer, therefore the velocity of the object in motion is proportional to the magnitude of its recorded fringe shift.

*Note: see Appendix A for a derivation of the above approximation.

## 2.2 Magnitude of the Fringe Shift

The interference of two beams of light can be found by computing the intensity of the total electric field. For electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$, the total electric field is the superposition of the two individual beams, where

$$\mathbf{E}_1 = \mathbf{E}_{01} e^{i(\mathbf{k}_1 \cdot \mathbf{r} - \omega t + \theta_1)}$$
$$\mathbf{E}_2 = \mathbf{E}_{02} e^{i(\mathbf{k}_2 \cdot \mathbf{r} - \omega t + \theta_2)}$$

Thus the total electric field is given by:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = [\mathbf{E}_{01} e^{i(\mathbf{k}_1 \cdot \mathbf{r} + \theta_1)} + \mathbf{E}_{02} e^{i(\mathbf{k}_2 \cdot \mathbf{r} + \theta_2)}] e^{-i\omega t}$$

In the above equations, $\mathbf{k}_1$ and $\mathbf{k}_2$ are the wavevectors of the respective electric fields and are not necessarily the same, depending on the medium through which each beam passes. ($k = n\frac{\omega}{c}$, where $n$ is the index of refraction of the medium; so, differing indices of refraction equates to differing wavevectors.) $\omega$ is the optical frequency of the light, and it is assumed that this term is equal for both beams, as they are both from the same light source. It is also assumed that the polarization of both components are the same, otherwise there would not be intereference between the two as they would be polarized in different planes. $\theta_1$ and $\theta_2$ are the initial phase shifts for each electric field, respectively, so the phase for each component of the total electric field is given by:

$$\phi_1 = \mathbf{k}_1 \cdot \mathbf{r} - \omega t + \theta_1$$
$$\phi_2 = \mathbf{k}_2 \cdot \mathbf{r} - \omega t + \theta_2$$

The intensity of light is equal to the square of the modulus of the total electric field:

$$\begin{aligned} I &= |\mathbf{E}|^2 = \mathbf{E} \cdot \mathbf{E}^* \\ &= (\mathbf{E}_1 + \mathbf{E}_2) \cdot (\mathbf{E}_1^* + \mathbf{E}_2^*) \\ &= |\mathbf{E}_1|^2 + |\mathbf{E}_2|^2 + (\mathbf{E}_1 \cdot \mathbf{E}_2^*) + (\mathbf{E}_2 \cdot \mathbf{E}_1^*) \\ &= I_1 + I_2 + (\mathbf{E}_1 \cdot \mathbf{E}_2^*) + (\mathbf{E}_2 \cdot \mathbf{E}_1^*) \end{aligned}$$

Considering the complex terms independently:

$$\mathbf{E}_1 \cdot \mathbf{E}_2^* = (\mathbf{E}_{01} \cdot \mathbf{E}_{02}^*) e^{i[(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r} + (\theta_1 - \theta_2)]} = \sqrt{I_1 I_2} e^{i[(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r} + (\theta_1 - \theta_2)]}$$

$$\mathbf{E}_2 \cdot \mathbf{E}_1^* = (\mathbf{E}_{02} \cdot \mathbf{E}_{01}^*) e^{-i[(\mathbf{k}_1 - \mathbf{k}_2)\cdot \mathbf{r} + (\theta_1 - \theta_2)]} = \sqrt{I_2 I_1} e^{-i[(\mathbf{k}_1 - \mathbf{k}_2)\cdot \mathbf{r} + (\theta_1 - \theta_2)]}$$

Hence:

$$I = I_1 + I_2 + \sqrt{I_1 I_2}(e^{i[(\mathbf{k}_1 - \mathbf{k}_2)\cdot \mathbf{r} + (\theta_1 - \theta_2)]} + e^{-i[(\mathbf{k}_1 - \mathbf{k}_2)\cdot \mathbf{r} + (\theta_1 - \theta_2)]})$$

Euler's Formula can be used to simplify the above equation, given that

$$\cos \delta = \frac{1}{2}(e^{i\delta} + e^{-i\delta}).$$

So the total intensity is equal to

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \delta,$$

where $\delta$ is the difference in phase between the two and is equal to

$$\begin{aligned} \delta &= (\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r} + (\theta_1 - \theta_2) \\ &= \phi_1 - \phi_2 = \Delta\phi. \end{aligned}$$

The resulting interference between the two waves is captured by the $2\sqrt{I_1 I_2} \cos \delta$ term in the equation for the total intensity. Thus, the magnitude of the fringe shift recorded by the streak camera can be mathematically expressed as the phase difference between the two beams.

## 2.3 Fourier Transform & Phase Extraction

The generalized real-valued function for an interferometric fringe pattern is given as follows:

$$g(x,t) = a(x,t) + b(x,t) \cos[2\pi f_0 x + \delta_0 + \phi(x,t)]$$

This equation is based on the derivation outlined in Taked et al.[9], (including a temporal dependence) and further studied in Celliers et al.[6]. Equating the above equation with the equation for the total intensity, one can see that $f(x,t) = I$, $a(x,t) = I_1 + I_2$, $b(x,t) = 2\sqrt{I_1 I_2}$, and $\cos[2\pi f_0 x + \delta_0 + \phi(x,t)] = \cos \delta$. In other words, $a(x,t)$ represents the background intensity, which is assumed to vary slowly in time; $b(x,t)$ represents the fringe amplitude, or the intensity of the interference fringes; $2\pi f_0 x + \delta_0$ represents the linear phase ramp of the background fringe pattern, which contains information about the frequency $f_0$ of the carrier wave and its initial phase offset $\delta_0$; and $\phi(x,t)$ represents the modulation in the phase and contains the information needed to extract the velocity of the object in motion.

Again using Euler's Formula – but this time in reverse – the above equation can be decomposed into an exponential and its complex conjugate. If we define a function $c(x,t)$ as

$$c(x,t) = \frac{1}{2} b(x,t) e^{i\delta_0} e^{i\phi(x,t)},$$

then it follows that $g(x,t)$ can be expressed as

$$g(x,t) = a(x,t) + c(x,t) e^{i(2\pi f_0 x)} + c^*(x,t) e^{-i(2\pi f_0 x)},$$

where $^*$ denotes the complex conjugate. To express the above equation in frequency space, a spatial Fourier transform can be applied at a fixed point in time, yielding the following:

$$\begin{aligned} G(f,t) &= A(f,t) + \int_{-\infty}^{\infty} c(x,t) e^{i(2\pi f_0 x)} e^{-2\pi i f x} dx + \int_{-\infty}^{\infty} c^*(x,t) e^{-i(2\pi f_0 x)} e^{-2\pi i f x} dx \\ &= A(f,t) + \int_{-\infty}^{\infty} c(x,t) e^{-2\pi i (f - f_0) x} dx + \int_{-\infty}^{\infty} c^*(x,t) e^{-2\pi i (f + f_0) x} dx \end{aligned}$$

Or, in a more generalized form:

$$G(f,t) = A(f,t) + C(f - f_0, t) + C^*(f + f_0, t)$$

$G(f,t)$ is the spatial Fourier transform of $g(x,t)$ with respect to $x$. With the aid of a computer and a Fast Fourier Transform (FFT) algorithm, this process can be done at each point in time to produce a spectrogram of peaks in Fourier space, with one centered around zero ($A(f,t)$), and the other two on either side of zero at $f = f_0$ and $f = -f_0$, which correspond to $C(f - f_0, t)$ and $C^*(f + f_0, t)$, respectively.

By choosing the correct filter, the frequency peaks at $f = 0$ and $f = -f_0$ can be subtracted out of the spectrogram, and all that is left is $C(f - f_0, t)$, which can be translated to the origin by subtracting $f_0$ along the frequency axis to obtain $C(f,t)$.

An inverse Fourier transform of $C(f,t)$ with respect to $f$ can be taken to produce a complex valued function that contains only the necessary information for computing the desired phase information:

$$\begin{aligned} d(x,t) &= \int_{-\infty}^{\infty} C(f,t)e^{2\pi i f x} df \\ &= c(x,t)e^{2\pi i f_0 x} \\ &= \frac{1}{2}b(x,t)e^{i(2\pi f_0 x + \delta_0 + \phi(x,t))} \end{aligned}$$

From here, there are two different ways quoted in the literature in which the desired phase information can be extracted from the above equation. The first method includes simply taking the logarithm of the above equation:

$$\ln[d(x,t)] = \ln[\frac{1}{2}b(x,t)] + 2\pi i f_0 x + i\delta_0 + i\phi(x,t)$$

Now the phase function is completely separated from the rest of the terms and can be isolated by subtracting out the unwanted information. However, this phase is only determinate up to a factor of $2\pi$, therefore further analysis needs to be done to determine its absolute value. The other method of isolating the phase term includes finding the "wrapped" phase from the complex function $d(x,t)$. After taking the inverse Fourier transform, the function $d(x,t)$ can be separated into its real and imaginary components:

$$\begin{aligned} Re[d(x,t)] &= \frac{1}{2}b(x,t)\cos[2\pi f_0 x + \delta_0 + \phi(x,t)] \\ Im[d(x,t)] &= \frac{1}{2}b(x,t)\sin[2\pi f_0 x + \delta_0 + \phi(x,t)] \end{aligned}$$

From here, the wrapped phase can be extracted by taking the inverse tangent of the real and imaginary parts of $d(x,t)$:

$$\tan^{-1}\Big(\frac{Re[d(x,t)]}{Im[d(x,t)]}\Big) = W[2\pi f_0 x + \delta_0 + \phi(x,t)]$$

The wrapped phase function, denoted W, is bounded by the interval where the inverse tangent is defined, i.e. $(-\pi, \pi)$. Therefore, again, this function is only determinate up to a factor of $2\pi$, as it is discontinuous when the phase passes through odd multiples of $\pi$. In order to remove the discontinuities that occurr at every interval of $2\pi$ and extract the phase, the phase function must be unwrapped and the background subtracted out. Unwrapping the phase is carried out by first correcting for any discontinuity that is $\geq 2\pi$ for all points in space, and then doing the same for all points in time. However, for noisy data, this method does not work perfectly and some discontinuities may still exists in time or space, depending on the order in which the phase unwrapping is done (i.e. first in space and second in time, or vice versa).

## 2.4 Velocity per Fringe

A quantity known as the velocity per fringe, or VPF, is needed in order to calculate the free-surface velocity (FSV) of the object in motion. The magnitude of the fringe shift (i.e. the change in phase) is multiplied by the VPF to calculate the FSV. The equation for the VPF is given as:

$$VPF_0 = \frac{\lambda}{2\tau(1+\delta)},$$

where $\lambda$ is the wavelength of the Doppler shifted laser light, $\tau$ is the temporal delay between the two arms of the interferometer caused by the insertion of an etalon, and $\delta$ is a corrective term that accounts for the chromatic dispersion in the etalon. This equation was first derived by Barker and Hallenbach in 1970[1], and later extended to VISAR analyis in their 1972 paper[2] and expanded upon by Barker and Shuler in 1974[3] to include the $\delta$ term. Also derived in the previous papers is the equation for the temporal $\tau$ delay caused by the etalon:

$$\tau = \frac{2h}{c}\left(n - \frac{1}{n}\right),$$

where $h$ is the thicknes and $n$ is the index of refraction of the etalon, and $c$ is the speed of light. If the incident light does not enter the etalon orthogonal to its front edge, a corrective factor can be added to the denominator of the above equation to account for the angle of light path through the etalon[5]:

$$\tau = \frac{2h}{c \cdot \cos\left(\sin^{-1}\left(\frac{\sin\theta_h}{n}\right)\right)}\left(n - \frac{1}{n}\right)$$

The above equation for velocity per fringe $VPF_0$ only applies to free surfaces. When the sample is being viewed through some transparent viewing window, a corrective factor must be included in order to account for the change in refractive index when propagating through the material:

$$VPF = \frac{VPF_0}{1 + \frac{\Delta v}{v_0}},$$

where $\frac{\Delta v}{v_0}$ is the corrective factor that takes into account the index of refraction of the viewing window.

## 2.5 Velocity vs. Time

The end-goal of VISAR analysis is to produce a graph of Velocity vs. Time for a given point of the object in motion. The velocity of the object in motion is most commonly referred to as the particle velocity. When the velocity-per-fringe equation is applied to each spatial point on the graph of the unwrapped phase function, a plot is produced which contains information about the particle velocity at each location in space for any given point in time. This is known as the velocity map or final velocity field. By extracting a lineout for a single spatial point for a duration of time, the Velocity vs. Time can be plotted. This is what is known as the velocity history of the object of interest.

## 2.6 Internal Pressure

While technically outside of the scope of standard line-imaging VISAR analysis, the velocity history produced by VISAR analysis can allow for a calculation of the internal pressures produced within the object of interest as a result of the induced shock. There is, however, no analytical formula to calculate this pressure for every type of material that can be tested. All that exists are databases of empirical data that can be used to equate particle velocities to pressures for a given material and density using an equation of state. One such database is the LASL Shock Hugoniot Data book published by Los Alamos[8] (http://large.stanford.edu/publications/coal/references/docs/shd.pdf).

This database stemmed from the need for equation-of-state data for higher pressures that could not be determined through isothermal compressibilities. An experimental technique was developed at Los Alamos for determining the locus of single-shocked states within this higher pressure regime – known as the Hugoniot locus – and then using thermodynamic relations to calculate the resulting states. Shock velocity ($U_s$), mass velocity ($U_p$), and free-surface velocity ($U_{fs}$) are determined from the Hugoniot locus, and can be related to the material's pressure ($P$), volume ($V$), density ($\rho$), and internal energy ($E$) via the Rankine-Hugoniot relations that result from conservation of mass, momentum, and energy across the shock front:

- Mass conservation: $V/V_0 = (U_s - U_p)/U_s$
- Momentum conservation: $P - P_0 = \rho_0 U_s U_p$
- Energy conservation: $E - E_0 = \frac{1}{2}(P + P_0)(V_0 - V)$

The free-surface velocity is the sum of the velocity initially imparted to the material by the shock wave (i.e. the mass velocity) and the velocity that results from rarefaction waves that later pass through the material ($U_r$), returning it to its ambient state:

$$U_{fs} = U_p + U_r.$$

$U_r$ can be derived using the following equation:

$$U_r = \int_0^p -(\frac{\partial V}{\partial P})_s^{1/2} dP,$$

where the integration is taken along an isentrope.

# 3 Appendix A

## 3.1 Doppler Shift Approximation

The equation for the relativistic Doppler shift can be significantly simplified if we use a binomial expansion and only keep the first two terms. For a function of the form $(1+\beta)^\alpha$, as long as $\beta$ is very small (i.e. $\beta << 1$), the following approximation can be made:

$$(1+\beta)^\alpha \approx 1 + \alpha\beta$$

For the relativistic Doppler shift, it is generally the case that $\beta = v/c << 1$, since the velocity of the object in motion is much smaller than the speed of light. The equation can be simplified in the following way:

$$\frac{\lambda}{\lambda_0} = \sqrt{\frac{1-v/c}{1+v/c}} = \frac{(1-v/c)^{1/2}}{(1+v/c)^{1/2}} = (1-v/c)^{1/2}(1+v/c)^{-1/2}$$

From here, the binomial approximation can be used to give:

$$\frac{\lambda}{\lambda_0} \approx (1 - \frac{1}{2}\frac{v}{c})(1 - \frac{1}{2}\frac{v}{c}) = 1 - \frac{v}{c} + \frac{1}{4}\frac{v^2}{c^2}$$

The last term can be discarded due to $v^2/c^2 \approx 0$, so it follows that the Doppler shifted wavelength is directly proportional to the velocity of the object in motion:

$$\frac{\lambda}{\lambda_0} \approx 1 - v/c \Rightarrow \lambda \propto v$$

# References


[1] L_M Barker and RE Hollenbach. Shock-wave studies of pmma, fused silica, and sapphire. *Journal of Applied Physics*, 41(10):4208–4226, 1970.

[2] LM Barker and RE Hollenbach. Laser interferometer for measuring high velocities of any reflecting surface. *Journal of Applied Physics*, 43(11):4669–4675, 1972.

[3] LM Barker and KW Schuler. Correction to the velocity-per-fringe relationship for the visar interferometer. *Journal of Applied Physics*, 45(8):3692–3693, 1974.

[4] DD Bloomquist and SA Sheffield. Optically recording interferometer for velocity measurements with subnanosecond resolution. *Journal of Applied Physics*, 54(4):1717–1722, 1983.

[5] CA Bolme and KJ Ramos. Line-imaging velocimetry for observing spatially heterogeneous mechanical and chemical responses in plastic bonded explosives during impact. *Review of Scientific Instruments*, 84(8):083903, 2013.

[6] PM Celliers, DK Bradley, GW Collins, DG Hicks, TR Boehly, and WJ Armstrong. Line-imaging velocimeter for shock diagnostics at the omega laser facility. *Review of scientific instruments*, 75(11):4916–4929, 2004.

[7] W Hemsing, A Mathews, R Warnes, M George, and G Whittemore. *Shock Compression of Condensed Matter-1991: Proceedings of the American Physical Society Topical Conference Held in Williamsburg, Virginia, June 17-20, 1991.* Elsevier, 1992.

[8] Stanley P Marsh. *LASL shock Hugoniot data*, volume 5. Univ of California Press, 1980.

[9] Mitsuo Takeda, Hideki Ina, and Seiji Kobayashi. Fourier-transform method of fringe-pattern analysis for computer-based topography and interferometry. *JosA*, 72(1):156–160, 1982.