Maximum Likelihood Analysis of LAT Data

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Source Model and Instrument Response

The total source model is the sum of contributions from individual point-like and diffuse sources:

$$S(\varepsilon, \hat{p}) = \sum_i S_i(\varepsilon, \hat{p}), \tag{1}$$

where ε is the true energy of the photon and $\hat{p}$ is the true direction on the sky. For a point source i, the spatial and spectral part can be factored

$$S_i(\varepsilon, \hat{p}) = \tilde{S}_i(\varepsilon)\delta(\hat{p} - \hat{p}_i). \tag{2}$$

The instrument response is typically factored into three components:

$$R(\varepsilon', \hat{p}'; \varepsilon, \hat{p}) = A(\varepsilon, \hat{p})P(\hat{p}'; \varepsilon, \hat{p})D(\varepsilon'; \varepsilon, \hat{p}). \tag{3}$$

Here ε' and $\hat{p}'$ are the measured energy and direction of the photon, repectively. $P(\hat{p}'; \varepsilon, \hat{p})$ is the point spread function; and $D(\varepsilon'; \varepsilon, \hat{p})$ is the energy dispersion; both functions are pdfs. The effective area $A(\varepsilon, \hat{p})$ is the cross-section of the LAT for detecting an incident photon with $(\varepsilon, \hat{p})$.

Binned Likelihood

For data binned in $(\varepsilon', \hat{p}')$, the likelihood is

$$\mathcal{L} = \prod_j \frac{\theta_j^{n_j} e^{-\theta_j}}{n_j!}, \tag{4}$$

where n_j is the number of events in bin j, and θ_j is the predicted number of events lying in that bin given the model:

$$\theta_j = \int_j d\varepsilon' d\hat{p}' \int_{\rm SR} d\varepsilon\, d\hat{p}\, R(\varepsilon', \hat{p}'; \varepsilon, \hat{p})\, S(\varepsilon, \hat{p}) \tag{5}$$

Here $\int_j d\varepsilon' d\hat{p}'$ indicates the integral over the jth bin, and $\int_{\rm SR} d\varepsilon d\hat{p}$ is the integral over the "source region", which in theory, covers the whole sky and all energies.

Unbinned Likelihood

The log-likelihood for the binned case:

$$\begin{aligned} \log \mathcal{L} &= \sum_j (n_j \log \theta_j - \theta_j) - \cancel{\log n_j!} && (6) \\ &= \sum_j n_j \log \theta_j - N_{\rm pred} && (7) \end{aligned}$$

In the limit of very small bins, where we have 0 or 1 event per bin,

$$\theta_j = \delta\varepsilon' \, \delta\hat{p}' \int_{\rm SR} d\varepsilon \, d\hat{p} \, R({\varepsilon'}_j, \hat{p}'_j; \varepsilon, \hat{p}) \, S(\varepsilon, \hat{p}). \tag{8}$$

Here j labels each detected photon, and we obtain the expression for our unbinned likelihood calculation:

$$\log \mathcal{L} = \sum_j \log \left(\int_{\rm SR} d\varepsilon \, d\hat{p} \, R({\varepsilon'}_j, \hat{p}'_j; \varepsilon, \hat{p}) S(\varepsilon, \hat{p}) \right) - \cancel{N_{\rm obs} \log(\delta\varepsilon' \, \delta\hat{p}')} - N_{\rm pred} \tag{9}$$

Parameter Estimation and Errors

Point sources are typically fit at known positions (ascertained by observations at other frequencies) with relatively simple spectral models, e.g., power-laws:

$$\tilde{S}(\varepsilon) = N_0 \varepsilon^{-\Gamma} \tag{10}$$

No matter what the spectral model, there will always be a normalization parameter N_0 that will be zero in null hypothesis. In any observation, in addition to discrete components, there will be Galactic and extragalactic diffuse components and instrumental background. The former two must also be modeled (fit) with some set of parameters.

For error estimates, we use the interval corresponding to

$$\Delta \log \mathcal{L} = 1/2 \tag{11}$$

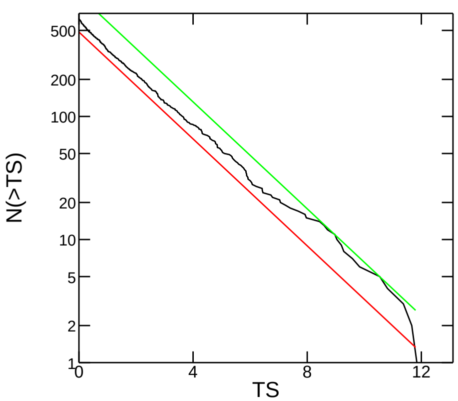
to give the "1-sigma" errors.

Source Significance and Null Distributions

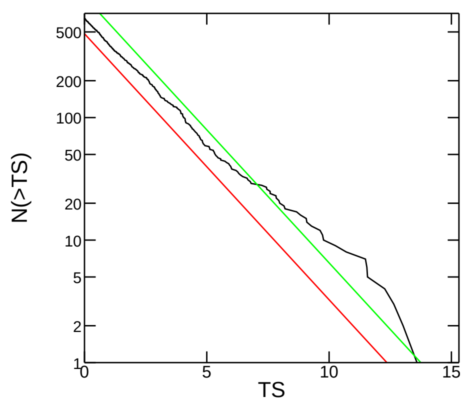
For ascertaining source significance, we use

$$T_s = 2(\log \mathcal{L} - \log \mathcal{L}_0) \tag{12}$$

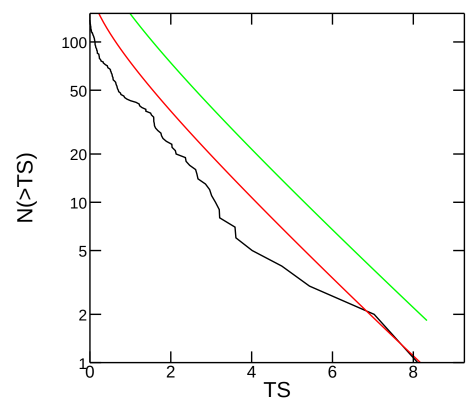
Since our model for the source spectrum has a normalization factor that is zero in the null case and cannot go below zero, we violate the regularity conditions that would otherwise allow us to use Wilks' theorem (Protassov et al 2002; Mattox et al. 1996). Following Mattox, we hoped that $\chi^2_0/2 + \chi^2_n/2$ would give us the desired reference distribution for determining p-values. Numerical simulations show that this is not the case:



Unbinned, $n = 2$



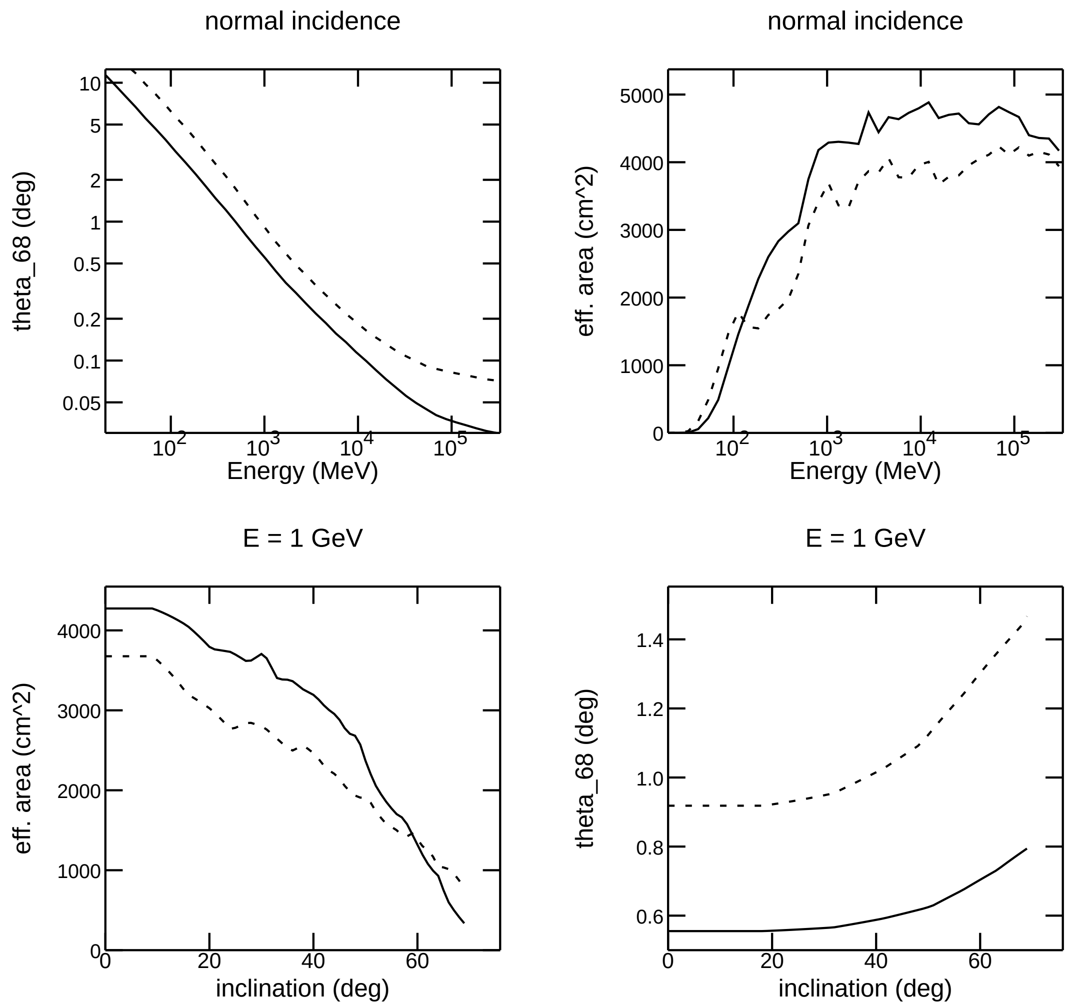
Binned, $n = 2$



Unbinned, $n = 1$

Red curves are $\chi^2_n/2$, and green curves are χ^2_n.

Instrument Response and Punzi Problem



The instrument response is a strong function of energy and inclination wrt the instrument z-axis.

The full log-likelihood (without constant terms)

$$\log \mathcal{L} = \sum_j \log \left(\int_{\mathrm{SR}} d\varepsilon \, d\hat{p} \, A(\varepsilon, \hat{p}) P(\hat{p}'_j; \varepsilon, \hat{p}) D({\varepsilon'}_j; \varepsilon, \hat{p}) S(\varepsilon, \hat{p}) \right) - N_{\mathrm{pred}} \quad (13)$$

$$= \sum_j \log \left(\sum_i \left[\int_{\mathrm{SR}} d\varepsilon \, A(\varepsilon, \hat{p}_i) P(\hat{p}'_j; \varepsilon, \hat{p}_i) D({\varepsilon'}_j; \varepsilon, \hat{p}_i) \tilde{S}_i(\varepsilon) \right] + \text{Diffuse terms} \right) - N_{\mathrm{pred}} \quad (14)$$

includes the "Punzi" factors for the probability that a given detected photon will have been measured with a certain resolution. However, in practice, we neglect the effects of $D(\varepsilon'; \varepsilon, \hat{p})$ ($\Delta\varepsilon/\varepsilon \approx 0.1$) so there will be some Punzi bias. Perhaps we see this for weak, soft sources that seem to get confused with extragalactic diffuse emission.