Maximum Likelihood:

Statistics in photon counting experiments

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Questions in γ-ray astronomy

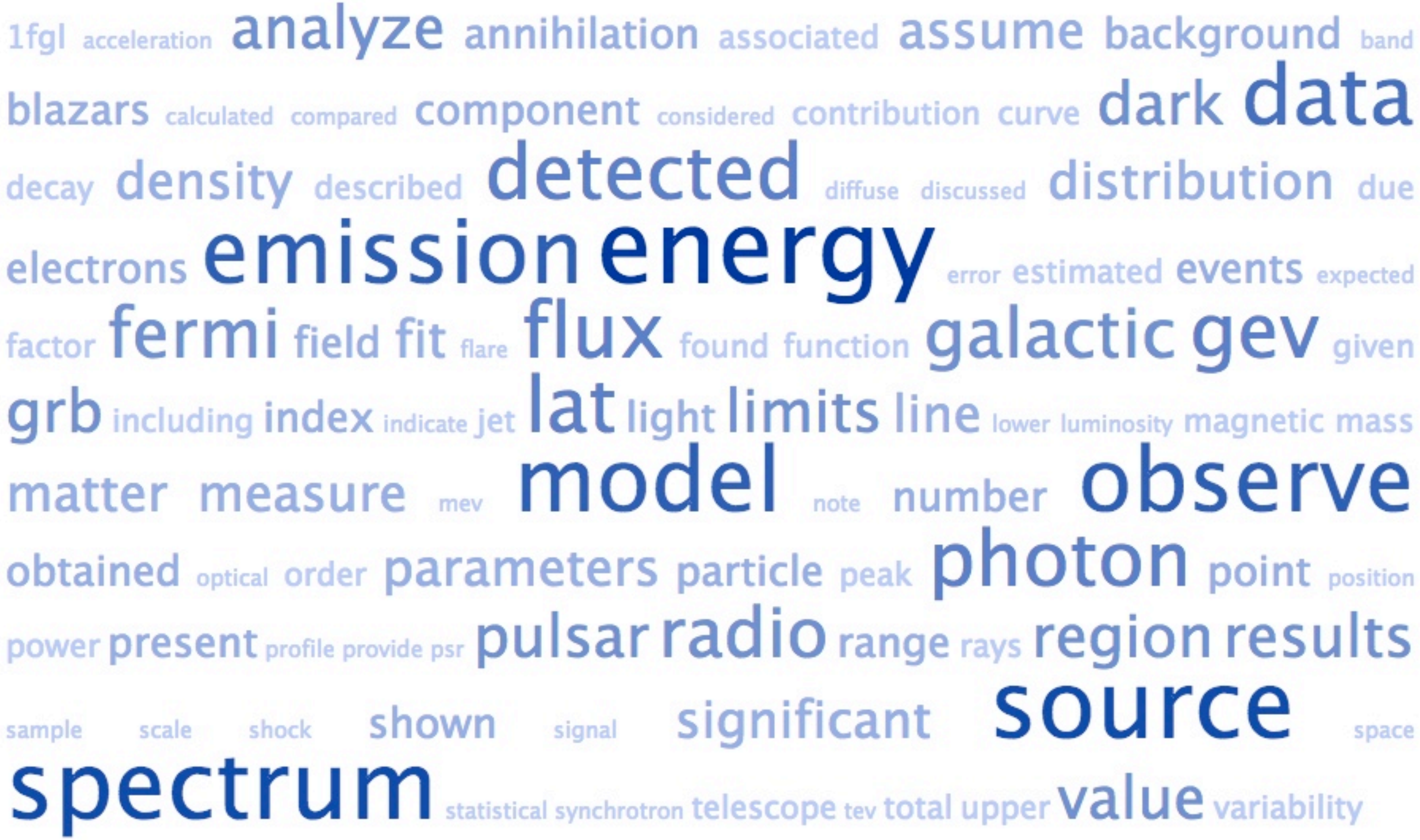
- Is a source significantly detected?
- If so, what is its flux?
- If not, what is upper limit on the flux?
- What kind of spectrum does it have?
- What is its spectral index?
- What is its location in the sky?
- What are the errors on these values?
- Is the source variable?

Questions in DM astrophysics

- Does Fermi detect γ-ray line emission from DM particle annihilation?
- With what significance?
- What is the energy of the line?
- What is the measurement error?
- What is the spatial distribution?
- What kind of systematic errors may be present?

LAT paper tag cloud

100 most frequently used words from 875 papers mentioning LAT and γ-ray on arXiv



1fgl acceleration analyze annihilation associated assume background band blazars calculated compared component considered contribution curve dark data decay density described detected diffuse discussed distribution due electrons emission energy error estimated events expected factor fermi field fit flare flux found function galactic gev given grb including index indicate jet lat light limits line lower luminosity magnetic mass matter measure mev model note number observe obtained optical order parameters particle peak photon point position power present profile provide psr pulsar radio range rays region results sample scale shock shown signal significant source space spectrum statistical synchrotron telescope tev total upper value variability

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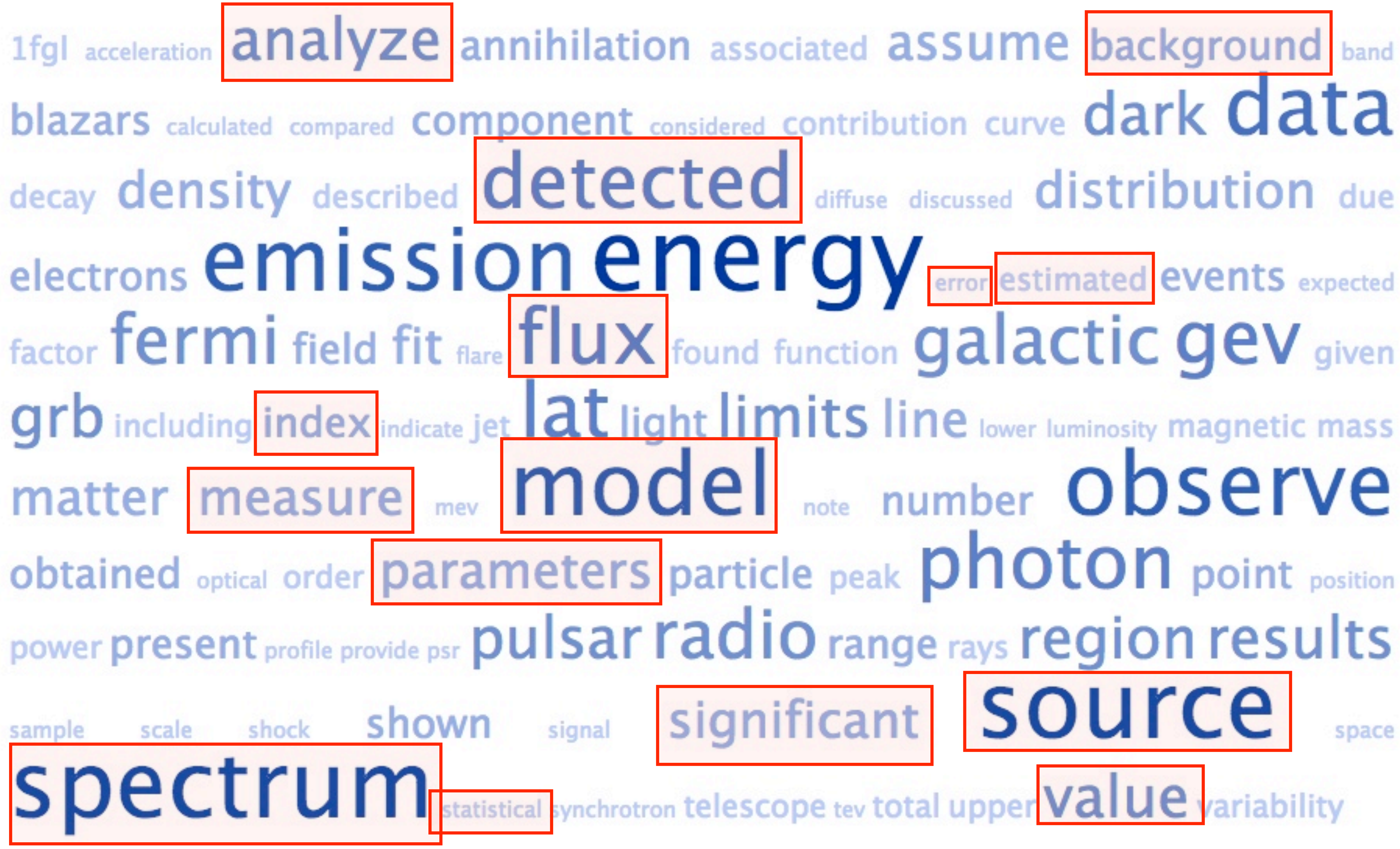

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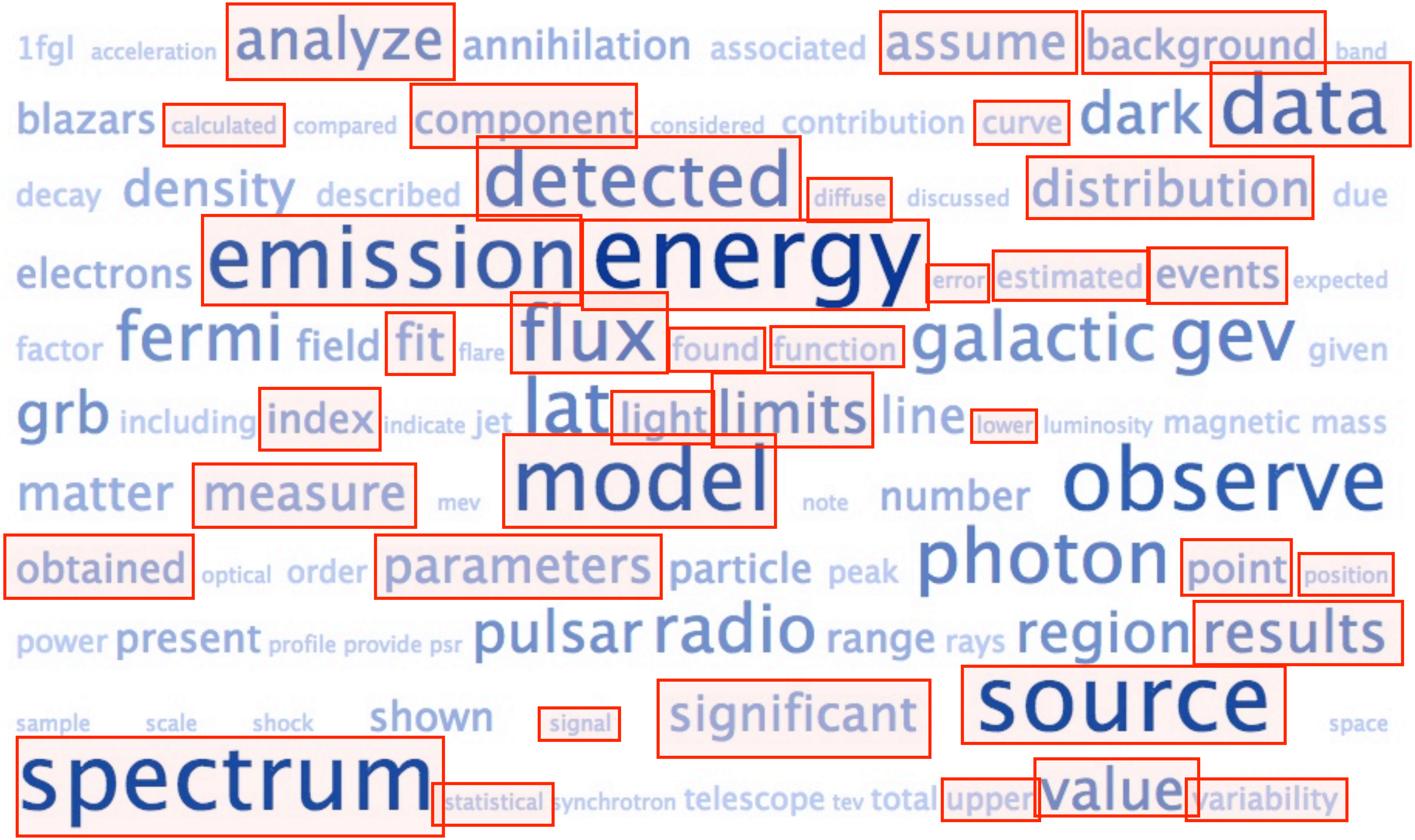

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Hypothesis testing	• Is a source significantly detected?
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Maximum likelihood estimation

- Given a set of observed data:
- ... produce a model that *accurately* describes the data, including parameters that we wish to estimate,
- ... derive the probability (density) for the data given the model (PDF),
- ... treat this as a function of the model parameters (likelihood function), and
- ... maximize the likelihood with respect to the parameters - ML estimation.

Maximum likelihood basics

- Data: $X = \{x_i\} = \{x_1, x_2, ..., x_N\}$
- Model parameters: $\Theta = \{\theta_j\} = \{\theta_1, \theta_2, ..., \theta_M\}$
- Likelihood: $\mathcal{L}(\Theta|X) = P(X|\Theta)$
- Conditional probability rule for independent events: $P(A, B) \underset{\text{CPR}}{=} P(A)P(B|A) \underset{\text{Independence}}{=} P(A)P(B)$
- For independent data:

$$P(X|\Theta) = P(\{x_i\}|\Theta) = P(x_1|\Theta)P(x_2, .., x_N|\Theta) = \cdots$$

$$= P(x_1|\Theta)P(x_2|\Theta) \cdots P(x_N|\Theta) = \prod_i P(x_i|\Theta)$$

$$\mathcal{L}(\Theta|X) = \prod_i P(x_i|\Theta)$$

ML estimation (MLE)

- Parameters can be estimated by maximizing likelihood. Easier to work with log-likelihood:

$$\boxed{\ln \mathcal{L}(\Theta) = \ln \mathcal{L}(\Theta|X) = \sum_i \ln P(x_i|\Theta)}$$

- Estimates of $\{\hat{\theta}_k\}$ from solving simultaneous equations:

$$\boxed{\left.\frac{\partial \ln \mathcal{L}}{\partial \theta_j}\right|_{\{\hat{\theta}_k\}} = 0}$$

- For one parameter, if we have: $\mathcal{L}(\theta) \sim e^{-\frac{(\theta-\hat{\theta})^2}{2\sigma_\theta^2}}$ then: $\left.\frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2}\right|_{\hat{\theta}} = -\frac{1}{\sigma_\theta^2}$

  Gaussian approximation

  so 2nd derivative is related to "errors"

Why maximum likelihood...

...rather than some ad-hoc estimation method?

- ML framework provides a “cookbook” through which problems can be solved.
  In other methods ad-hoc choices may have to be made.
- ML provides unbiased, minimum variance estimate as sample size increases.
  Same may not be case for ad-hoc methods.
- Asymptotically Gaussian: evaluation of confidence bounds & hypothesis testing.
- Well studied in the literature.
- Starting point for Bayesian analysis.

MLE example 1:

χ^2 fit of constant - I

- Data: independent measurements of flux of some source with errors - (x_i, σ_i)
- Model: all measurements are of a constant flux F with Gaussian errors.
- Probabilities: $P(x_i|F) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(x_i - F)^2}{2\sigma_i^2}}$
- Log likelihood:

$$\ln \mathcal{L}(F) = -\sum \frac{(x_i - F)^2}{2\sigma_i^2} - \sum \ln \sigma_i - \frac{N}{2} \ln 2\pi$$

MLE example 1:

χ^2 fit of constant - II

- Log likelihood:

Constant with respect to *F*

$$\ln \mathcal{L}(F) = -\sum \frac{(x_i - F)^2}{2\sigma_i^2} \cancel{\boxed{- \sum \ln \sigma_i - \frac{N}{2} \ln 2\pi}}$$

- Maximize for MLE of F:

$$\frac{\partial \ln \mathcal{L}}{\partial F} = \sum \frac{x_i - F}{\sigma_i^2} = 0 \Longrightarrow \boxed{\hat{F} = \frac{\sum x_i/\sigma_i^2}{\sum 1/\sigma_i^2}}$$

- Curvature gives “error” on *F*:

$$\frac{1}{\sigma_F^2} = - \left.\frac{\partial^2 \ln \mathcal{L}}{\partial F^2}\right|_{\hat{F}} = \sum \frac{1}{\sigma_i^2} \Longrightarrow \boxed{\sigma_F = \frac{1}{\sqrt{\sum 1/\sigma_i^2}}}$$

MLE example 2:

Event counting experiment

- Experiment detects *n* events (e.g. γ rays)
- Model: Poisson process with mean of of λ:

$$P(x|\theta) \rightarrow P(n|\lambda) = \frac{\lambda^n e^{-\lambda}}{n!}$$

- Log likelihood: $\ln \mathcal{L}(\lambda) = n \ln \lambda - \lambda - \ln n!$
- ML estimate and error in Gaussian regime:

$$\frac{\partial \ln \mathcal{L}}{\partial \lambda} = \frac{n}{\lambda} - 1 \implies \hat{\lambda} = n$$

$$\sigma_\lambda^2 = - \left. \frac{\partial^2 \ln \mathcal{L}}{\partial \lambda^2} \right|_{\hat{\lambda}} = \frac{n}{\hat{\lambda}^2} \implies \sigma_\lambda^2 = n$$

Gaussian approximation

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$$P(x|\theta) \to P(n|\lambda) = \frac{\lambda^n e^{-\lambda}}{n!}$$

- Log likelihood: $\ln \mathcal{L}(\lambda) = \underbrace{n \ln \lambda}_{\text{Data cpt}} - \underbrace{\lambda}_{\text{Npred}} \overset{\text{Constant WRT } \lambda}{\cancel{- \ln n!}}$
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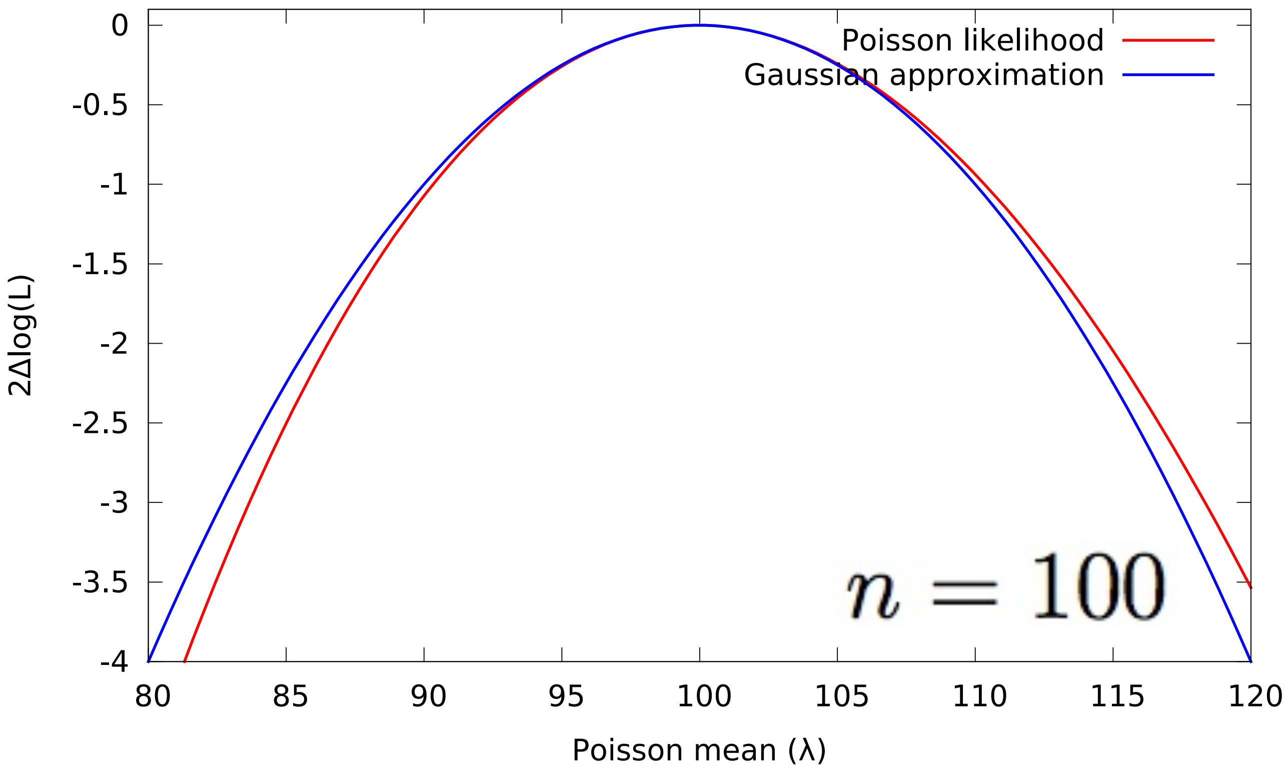
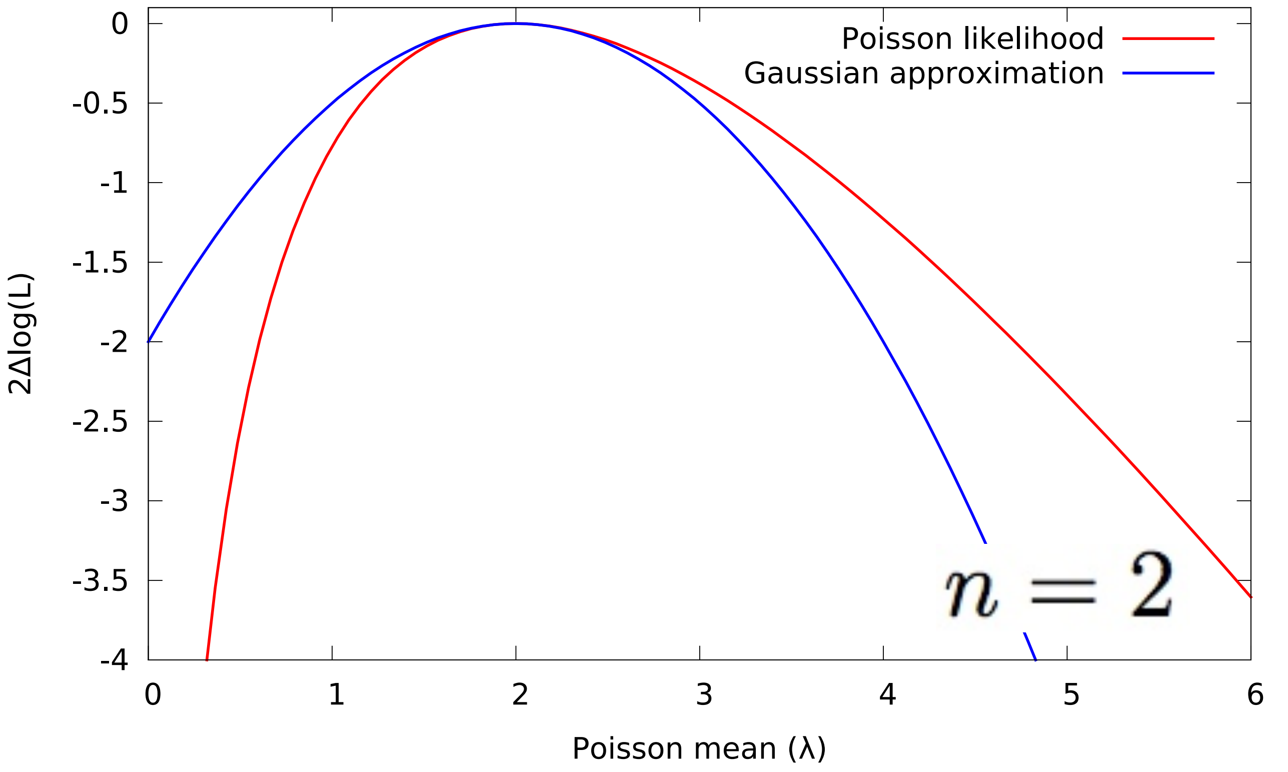
Gaussian approximation

MLE example 2:

Log-likelihood profiles





- Gaussian approximation is reasonable when *n* is “large enough”. In this case $\sigma_\lambda^2 = n$ is a good estimate of the “error”.

- If not, estimate errors by finding points where $2\ln\mathcal{L}(\lambda)$ decreases by 1.0 from maximum, i.e.,

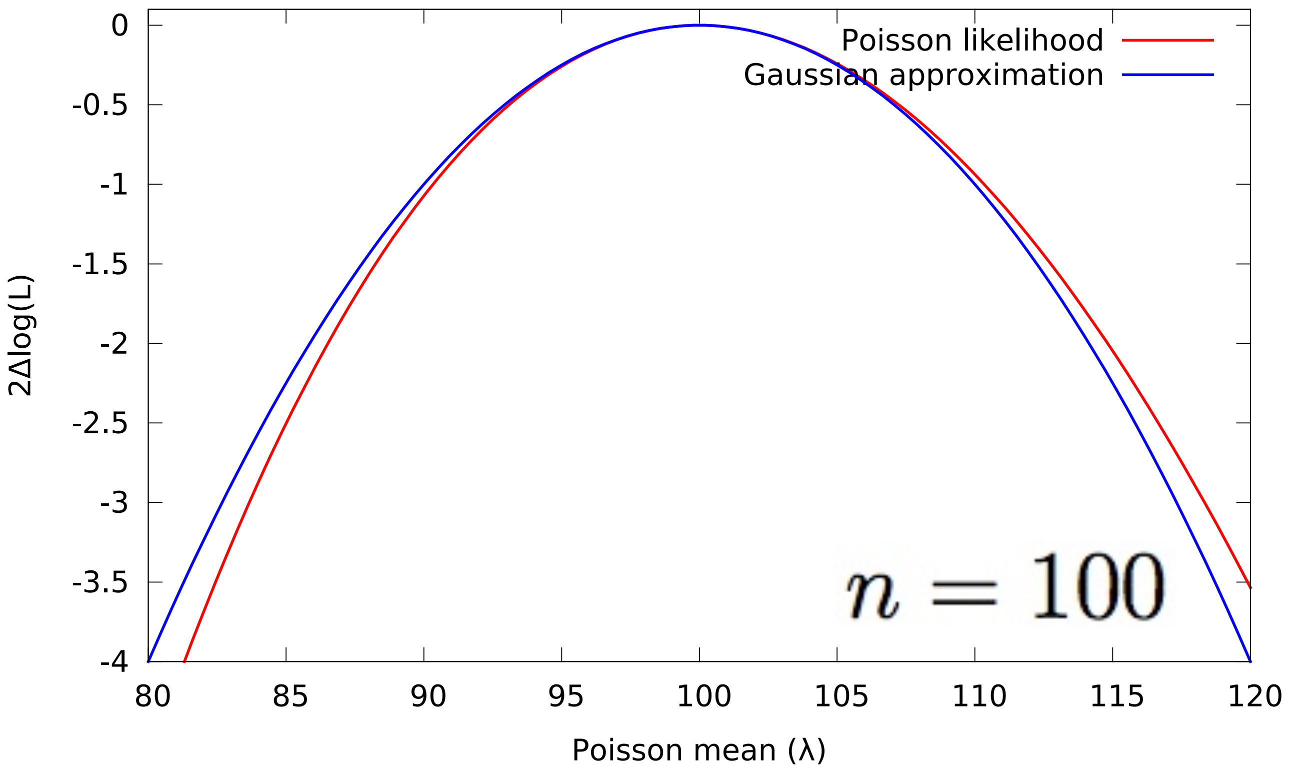
$$2\ln\mathcal{L}(\lambda) = 2\ln\mathcal{L}(\hat{\lambda}) - 1$$

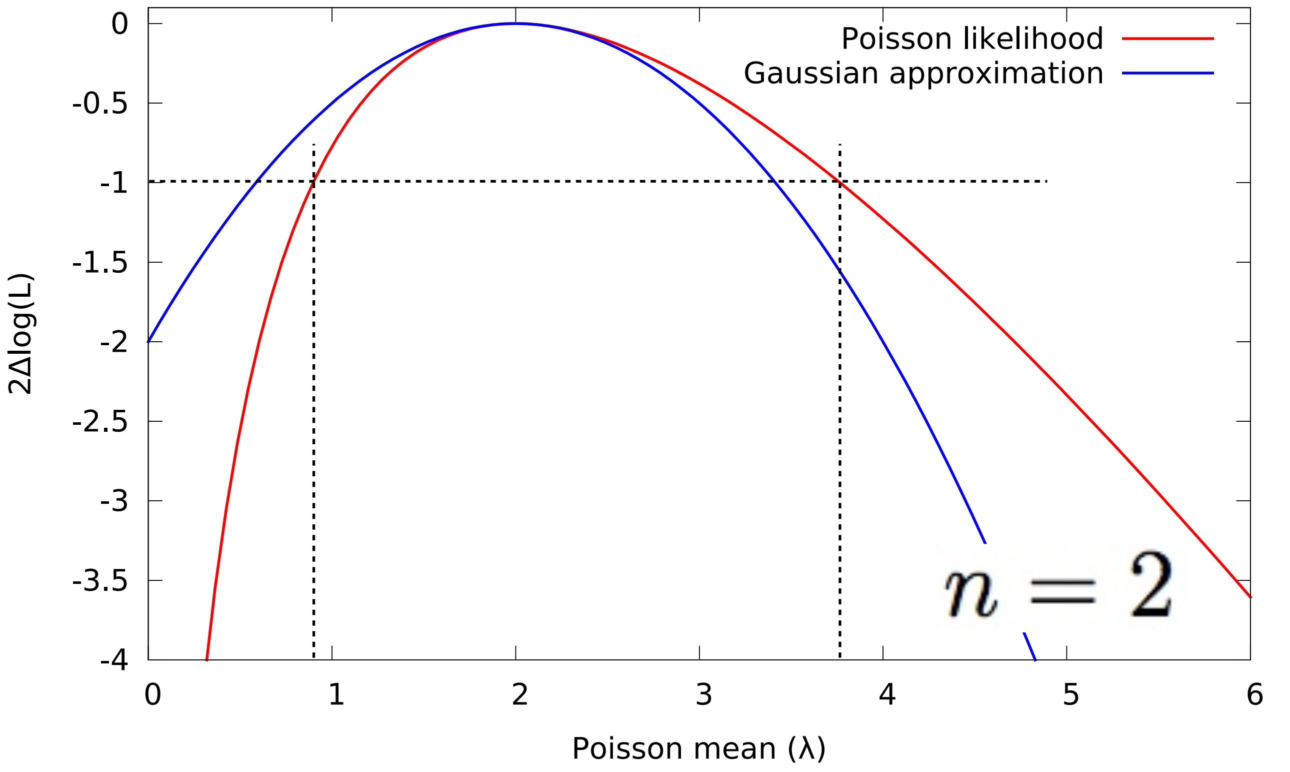
- *n=100* : $\hat{\lambda} = 100.0^{+10.33}_{-9.67}$

- *n=2* : $\hat{\lambda} = 2.0^{+1.77}_{-1.10}$

MLE example 2:

Log-likelihood profiles





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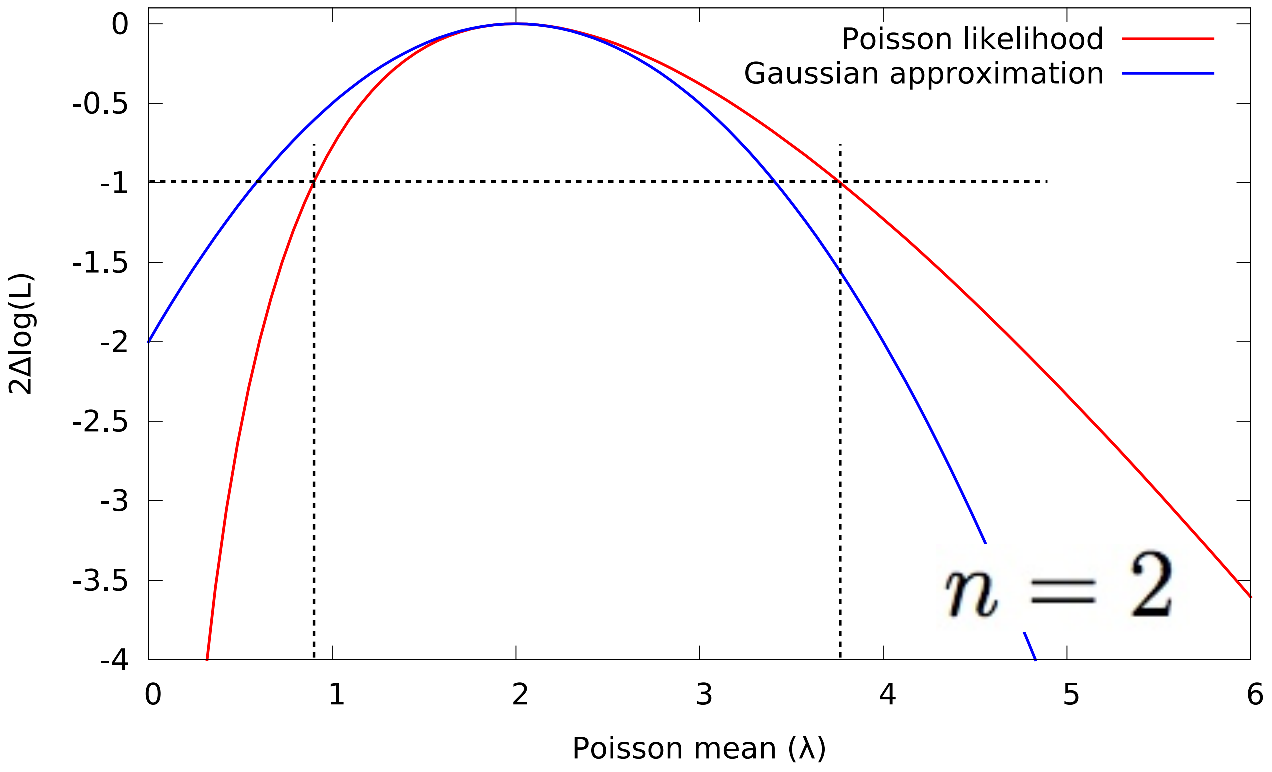
MLE example 2:

Log-likelihood profiles

```
# errors_poisson.py - 2013-05-07 SJF
# Evaluate the errors on the Poisson mean
import math, scipy.optimize
n_meas   = 2
logL     = lambda lam: n_meas*math.log(lam)-lam
opt_fn   = lambda lam: -logL(lam)
opt_res  = scipy.optimize.minimize(opt_fn, 1e-8)
lam_est  = opt_res.x[0]
logL_max = logL(lam_est)
root_fn  = lambda lam: 2.0*(logL(lam)-logL_max)+1.0
lam_lo   = scipy.optimize.brentq(root_fn, 1e-8, lam_est)
lam_hi   = scipy.optimize.brentq(root_fn, lam_est, 1e8)
print lam_est, lam_lo-lam_est, lam_hi-lam_est
```



$2 \ln \mathcal{L}(\lambda)$ decreases by 1.0 from maximum, i.e.,

$$2 \ln \mathcal{L}(\lambda) = 2 \ln \mathcal{L}(\hat{\lambda}) - 1$$

- *n=100* : $\hat{\lambda} = 100.0^{+10.33}_{-9.67}$
- *n=2* : $\hat{\lambda} = 2.0^{+1.77}_{-1.10}$

Hypothesis testing

- Compare likelihoods of two hypotheses to see which is better supported by the data.
- Likelihood-ratio test (LRT) & Wilks' theorem.
- Given a model with *N+M* parameters:

  $$\Theta = \{\theta_1, \ldots, \theta_N, \theta_{N+1}, \ldots, \theta_{N+M}\}$$

  where *N* have true values: $\theta_1^T, \ldots, \theta_N^T$
- Values of likelihood under two hypotheses:

$$\mathcal{L}_1 = \mathcal{L}(\hat{\theta}_1, \ldots, \hat{\theta}_N, \hat{\theta}_{N+1}, \ldots, \hat{\theta}_{N+M})$$

$$\mathcal{L}_0 = \mathcal{L}(\theta_1^T, \ldots, \theta_N^T, \hat{\theta}_{N+1}, \ldots, \hat{\theta}_{N+M})$$

- "Ratio" distributed as: $\boxed{2(\ln \mathcal{L}_1 - \ln \mathcal{L}_0) \sim \chi^2(N)}$

Terms and conditions apply

Why is that useful?

(We don't know the true values of any parameters!)

- We make an assumption about the model (*the null hypothesis*), in which the parameters have some presumed "true" values.

http://xkcd.com/892/

- Compute $\mathcal{L}_0$ from these values and $\mathcal{L}_1$ using MLE for all params.
- Hope to show that $2(\ln \mathcal{L}_1 - \ln \mathcal{L}_0)$ is so large that it is improbable from $\chi^2(N)$,
- and, hence, reject the null hypothesis. Usually cannot say hypothesis is true!

MLE example 3:

Source & Background

- Data: events detected in two independent "channels": $X = \{n_1, n_2\}$
- Model: Poisson process with...
  - Unknown "source" and "background":

$$\Theta = \{\theta_1, \theta_2\} = \{S, B\} \qquad \vec{\Theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} S \\ B \end{pmatrix}$$

  - Response matrix (presumed known) $\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$
  - Poisson means: $\vec{\lambda} = \mathbf{R}\vec{\Theta}$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} S \\ B \end{pmatrix}$$

MLE example 3:

MLE

- Log likelihood:

$$\begin{aligned}\ln\mathcal{L}(S,B) =& n_1\ln(r_{11}S + r_{12}B) + n_2\ln(r_{21}S + r_{22}B)\\ &- (r_{11}+r_{21})S - (r_{12}+r_{22})B + const\end{aligned}$$

- MLE: $\frac{\partial\ln\mathcal{L}}{\partial S} = \frac{\partial\ln\mathcal{L}}{\partial B} = 0 \implies \hat{\vec{\Theta}} = \mathbf{R}^{-1}\vec{n}$

$$\begin{pmatrix}\hat{S}\\ \hat{B}\end{pmatrix} = \frac{1}{r_{11}r_{22} - r_{12}r_{21}}\begin{pmatrix} r_{22} & -r_{12}\\ -r_{21} & r_{11}\end{pmatrix}\begin{pmatrix} n_1\\ n_2\end{pmatrix}$$

$$\ln\mathcal{L}_1 = \ln\mathcal{L}(\hat{S},\hat{B}) = n_1\ln n_1 + n_2\ln n_2 - (n_1+n_2)$$

- If likelihood: $\mathcal{L}(\vec{\Theta}) \sim e^{-\frac{1}{2}(\vec{\Theta}-\hat{\vec{\Theta}})^T\mathbf{\Sigma}^{-1}(\vec{\Theta}-\hat{\vec{\Theta}})}$

  “errors” are: $\left.\frac{\partial^2\ln\mathcal{L}}{\partial\theta_i\partial\theta_j}\right|_{\hat{\vec{\Theta}}} = -(\mathbf{\Sigma}^{-1})_{ij} = -\boldsymbol{\mathcal{I}}_{ij}$

Gaussian approximation

MLE example 3:

MLE

- Log likelihood:

$$\begin{aligned}\ln\mathcal{L}(S,B) = &\overbrace{n_1\ln(r_{11}S+r_{12}B)+n_2\ln(r_{21}S+r_{22}B)}^{\text{Data component}}\\ &\underbrace{-(r_{11}+r_{21})S-(r_{12}+r_{22})B}_{\text{Npred}}+\xcancel{const}\end{aligned}$$

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MLE

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Gaussian approximation

“errors” are: $\left.\dfrac{\partial^2\ln\mathcal{L}}{\partial\theta_i\partial\theta_j}\right|_{\hat{\vec{\Theta}}}=-(\underset{\substack{\uparrow\\ \text{Covariance matrix}}}{\boldsymbol{\Sigma}^{-1}})_{ij}=-\underset{\substack{\uparrow\\ \text{Fisher information matrix}}}{\mathcal{I}_{ij}}$

MLE example 3:

Covariances and errors

- Calculate Fisher information matrix and invert:

$$\boldsymbol{\mathcal{I}}_{ij} = - \left.\frac{\partial^2 \ln \mathcal{L}}{\partial\theta_i \partial\theta_j}\right|_{\hat{\vec{\Theta}}} \rightarrow \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_S^2 & \mathrm{cov}(S,B) \\ \mathrm{cov}(S,B) & \sigma_B^2 \end{pmatrix} = \boldsymbol{\mathcal{I}}^{-1}$$

- For our example we get:

$$\boldsymbol{\mathcal{I}} = \frac{1}{n_1 n_2} \begin{pmatrix} r_{21}^2 n_1 + r_{11}^2 n_2 & r_{21} r_{22} n_1 + r_{11} r_{12} n_2 \\ r_{21} r_{22} n_1 + r_{11} r_{12} n_2 & r_{22}^2 n_1 + r_{12}^2 n_2 \end{pmatrix}$$

$$\boldsymbol{\Sigma} = \frac{1}{\det(\mathbf{R})^2} \begin{pmatrix} r_{22}^2 n_1 + r_{12}^2 n_2 & -r_{21} r_{22} n_1 - r_{11} r_{12} n_2 \\ -r_{21} r_{22} n_1 - r_{11} r_{12} n_2 & r_{21}^2 n_1 + r_{11}^2 n_2 \end{pmatrix}$$

- In general parameters are correlated, but can choose set that is uncorrelated. Here they are $\{\lambda_1, \lambda_2\}$ giving $\hat{\lambda}_1 = n_1, \hat{\lambda}_2 = n_2, \boldsymbol{\Sigma}_\lambda = \mathrm{diag}(n_1, n_2)$

MLE example 3:

Source significance

- Null hypothesis: suppose $S = 0$, then:

$$\begin{aligned}\ln \mathcal{L}_0(B) &= \ln \mathcal{L}(S=0, B) \\ &= n_1 \ln r_{12} B + n_2 \ln r_{22} B - (r_{12} + r_{22}) B\end{aligned}$$

- MLE for *B* gives: $\frac{\partial \ln \mathcal{L}_0}{\partial B} = 0 \implies \hat{B}_0 = \frac{n_1 + n_2}{r_{12} + r_{22}}$

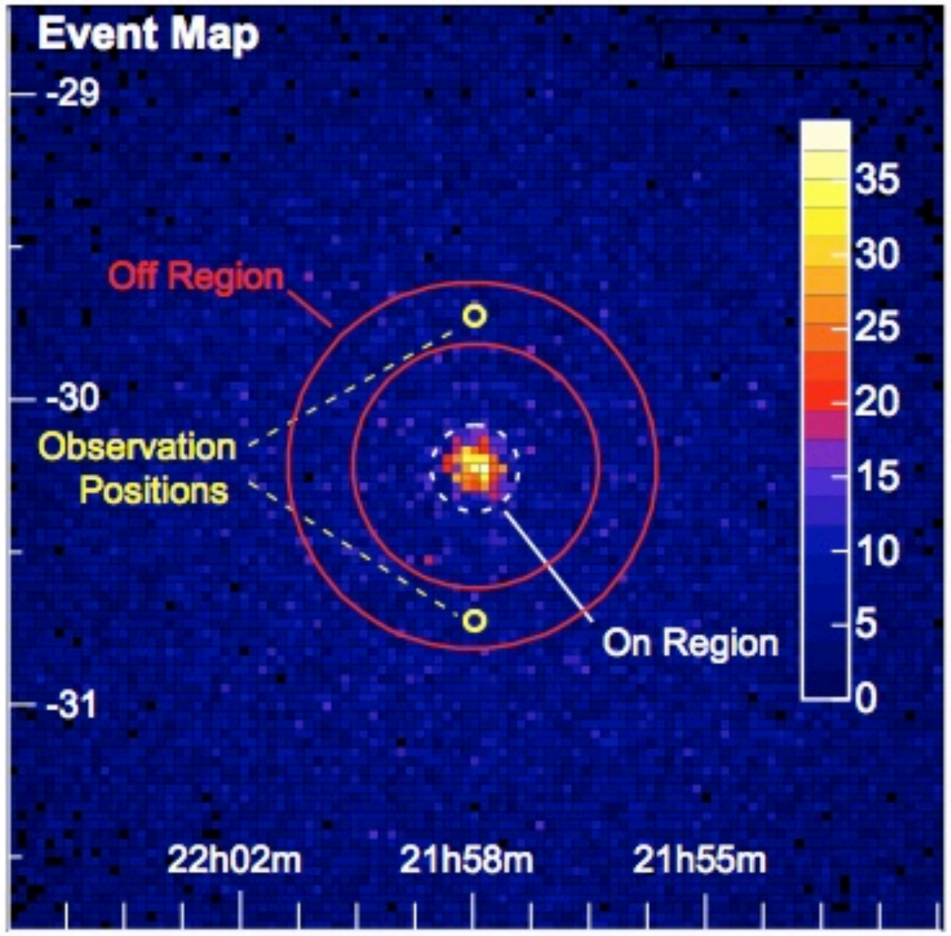
$$\begin{aligned}\ln \mathcal{L}_0 &= \ln \mathcal{L}_0(\hat{B}_0) \\ &= n_1 \ln \frac{r_{12}(n_1 + n_2)}{r_{12} + r_{22}} + n_2 \ln \frac{r_{22}(n_1 + n_2)}{r_{12} + r_{22}} - (n_1 + n_2)\end{aligned}$$

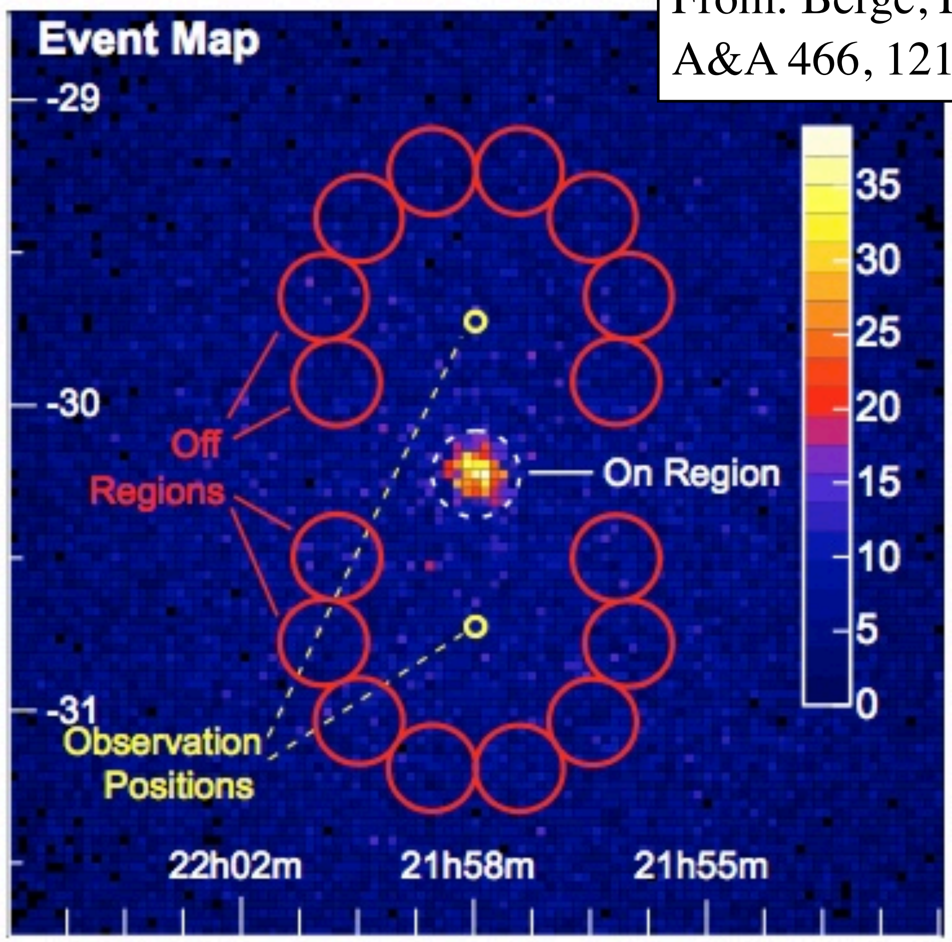
- Test statistic: $TS = 2(\ln \mathcal{L}_1 - \ln \mathcal{L}_0) \sim \chi^2(1)$

$$TS = 2\left[n_1 \ln \frac{(r_{12} + r_{22}) n_1}{r_{12}(n_1 + n_2)} + n_2 \ln \frac{(r_{12} + r_{22}) n_2}{r_{22}(n_1 + n_2)}\right]$$

MLE example 3:

On/Off problems



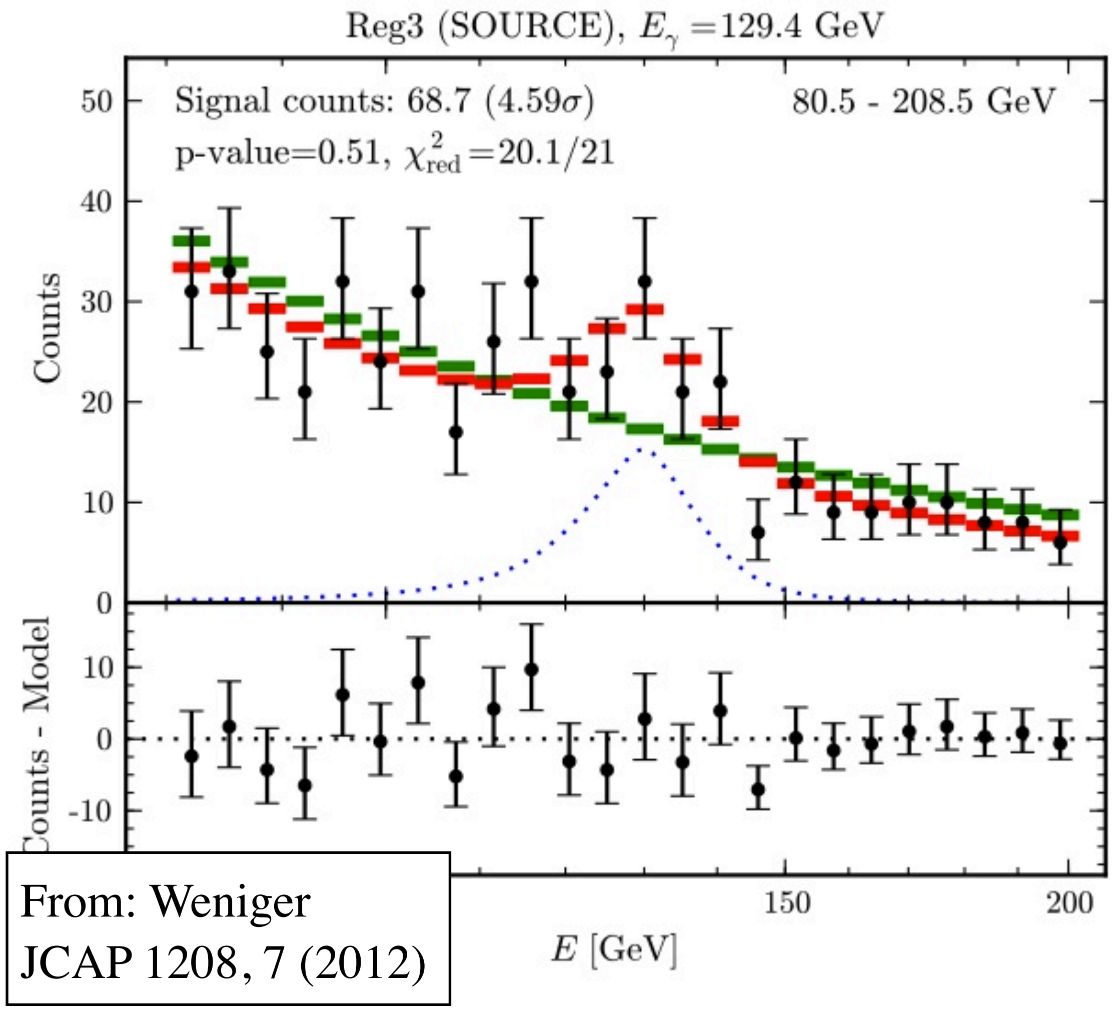


From: Berge, Funk, Hinton
A&A 466, 1219–1229 (2007)

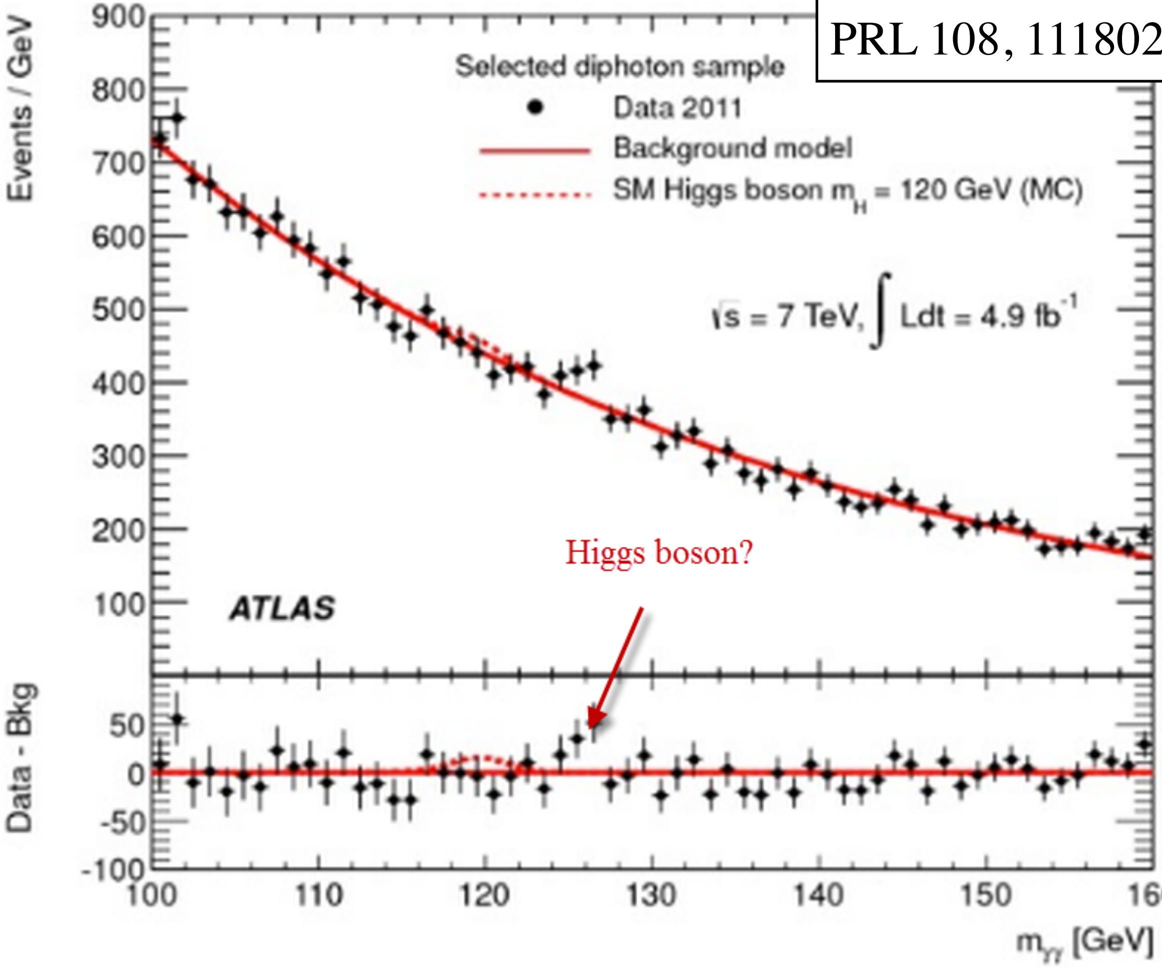
- VHE astronomy - gamma-ray sources and a background of cosmic rays.
- Problem - to evaluate flux of source and its statistical significance. Define on-source (source+background) and off-source (background) channels.

MLE example 3:

On/Off problems



From: Weniger
JCAP 1208, 7 (2012)



From: Aad et al.
PRL 108, 111802 (2012)

- Line searches - DM with Fermi, or Higgs with ATLAS.
- Problem - detect line signal on top of spectrum of background events. Define “on-source” and “off-source” regions. Must assume that spectrum of background is known or calculable.

MLE example 3:

On/Off problems

- General set of problems where:

$n_2 \rightarrow n_{off}$

$n_1 \rightarrow n_{on}$

$\lambda_2 \rightarrow \lambda_{off} = BT$

$\lambda_1 \rightarrow \lambda_{on} = (S + \alpha B)T$

- and where these are assumed to be known:

α - on to off-source background ratio

T - observation time (or other detector factors)

MLE example 3:

MLE for On/Off problems

- Then: $\mathbf{R} = T \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \quad \mathbf{R}^{-1} = \frac{1}{T} \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$

$$\ln \mathcal{L}(S,B) = n_{on} \ln[(S+\alpha B)T] + n_{off} \ln BT \\ - (S + (1+\alpha)B)T$$

- MLE & (co)variances of S and B are:

$$\hat{B} = \frac{1}{T} n_{off} \qquad \sigma_B^2 = \frac{1}{T^2} n_{off}$$

$$\hat{S} = \frac{1}{T}(n_{on} - \alpha n_{off}) \qquad \sigma_S^2 = \frac{1}{T^2}(n_{on} + \alpha^2 n_{off})$$

This is what you would expect!

$$\mathrm{cov}(\hat{S}, \hat{B}) = -\frac{1}{T^2} \alpha n_{off}$$

MLE example 3:

TS for On/Off problems

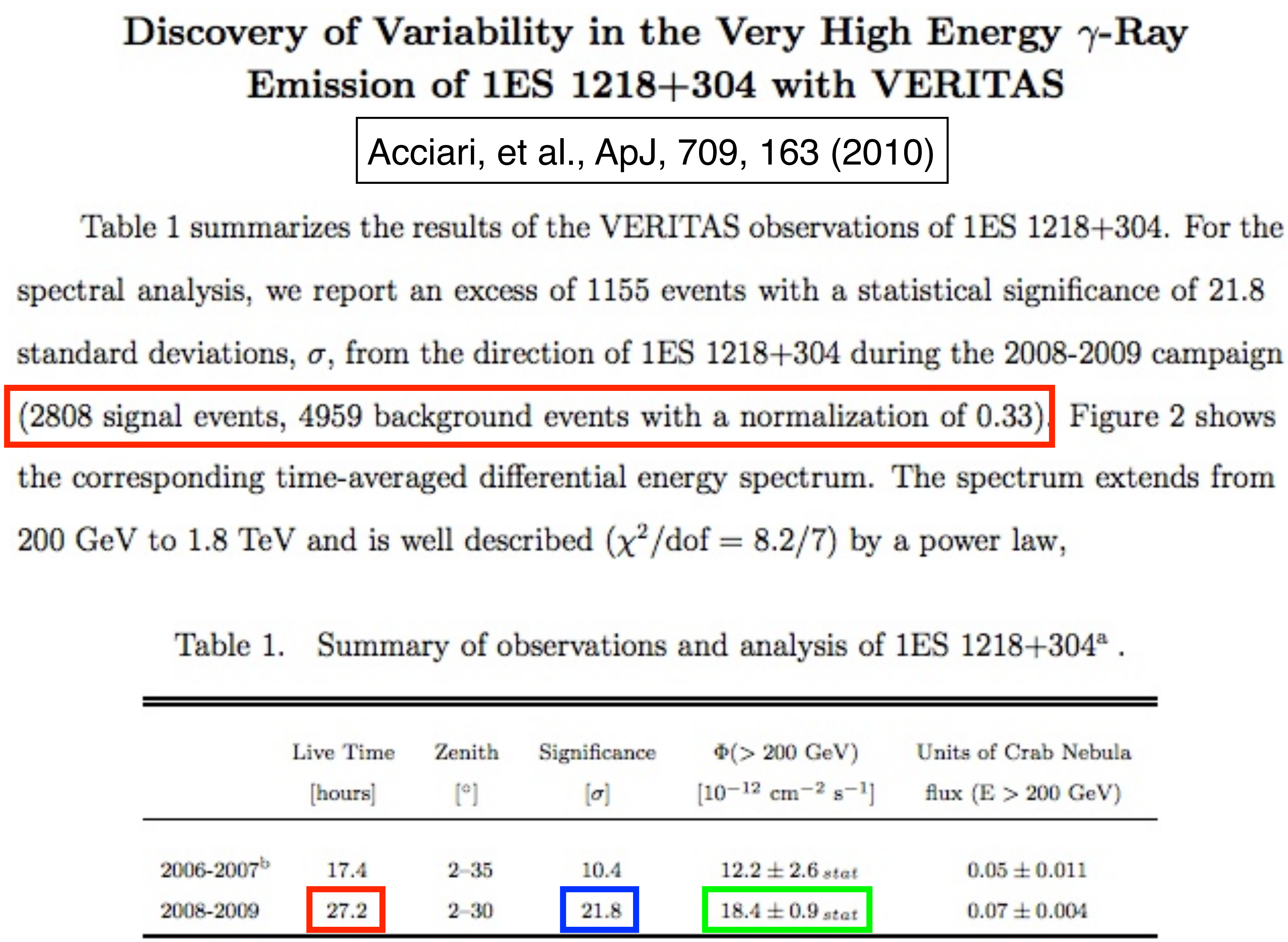
- Test statistic for source detection in On/Off problems is:

$$TS = 2\left[n_{on}\ln\frac{(1+\alpha)n_{on}}{\alpha(n_{on}+n_{off})} + n_{off}\ln\frac{(1+\alpha)n_{off}}{(n_{on}+n_{off})}\right]$$

- Significance is: $\sigma = \sqrt{TS}$
- This is the famous “Li & Ma” formula from: ApJ 272, 317 (1983) - 493 citations on ADS
- Probably, you wouldn’t arrive at this formula using ad hoc estimation methods
- P-values: `scipy.stats.chi2.sf(`*TS*`,1)`

MLE example 3:

Eg: 1ES1218+304 w/VERITAS



Discovery of Variability in the Very High Energy γ-Ray Emission of 1ES 1218+304 with VERITAS

Acciari, et al., ApJ, 709, 163 (2010)

Table 1 summarizes the results of the VERITAS observations of 1ES 1218+304. For the spectral analysis, we report an excess of 1155 events with a statistical significance of 21.8 standard deviations, σ, from the direction of 1ES 1218+304 during the 2008-2009 campaign (2808 signal events, 4959 background events with a normalization of 0.33). Figure 2 shows the corresponding time-averaged differential energy spectrum. The spectrum extends from 200 GeV to 1.8 TeV and is well described (χ^2/dof = 8.2/7) by a power law,

Table 1. Summary of observations and analysis of 1ES 1218+304[a] .

	Live Time [hours]	Zenith [°]	Significance [σ]	$\Phi(> 200$ GeV$)$ [10^{-12} cm^{-2} s^{-1}]	Units of Crab Nebula flux (E > 200 GeV)
2006-2007[b]	17.4	2–35	10.4	$12.2 \pm 2.6_{stat}$	0.05 ± 0.011
2008-2009	27.2	2–30	21.8	$18.4 \pm 0.9_{stat}$	0.07 ± 0.004

$$n_{off} = 4959$$
$$n_{on} = 2808$$
$$\alpha = 1/3$$
$$T = 27.2\,\mathrm{hr}$$

$$\hat{S} = 42.5\,\mathrm{hr}^{-1}$$
$$\sigma_S = 2.1\,\mathrm{hr}^{-1}$$

$$TS = 474.9$$

$$\sigma = 21.8$$

$$\mathrm{P-value} = 2.8 \times 10^{-105}$$

$$\sigma_{POE} = \frac{\hat{S}}{\sigma_S} = 19.9 \approx \frac{18.4}{0.9}$$

Ratio of value to error - used as “significance” before Li&Ma

MLE example 3:

Eg: 1ES1218+304 w/VERITAS

```
# lima.py - 2013-05-15 SJF
# Example of Li & Ma significance calculation
import math, scipy.stats

def ts_lima(non,noff,alpha):
    opa  = 1.0+alpha
    ntot = non+noff
    return 2.0*(non*math.log(opa*non/alpha/ntot) \
                + noff*math.log(opa*noff/ntot))


non    = 2808
noff   = 4959
alpha  = 1.0/3
T      = 27.2

S_hat  = (non - noff*alpha)/T
sig2_S = (non + noff*alpha**2)/T**2
ts     = ts_lima(non,noff,alpha)
signif = math.sqrt(ts)
Pval   = scipy.stats.chi2.sf(ts,1)

print S, math.sqrt(sig2_S), ts, signif, Pval
```

Ratio of value to error - used as "significance" before Li&Ma

MLE example 3:

Detectability / Sensitivity

- Interested in detectability of sources, i.e. sensitivity of instrument for given threshold.
- Consider “no fluctuations” case where:

$$n_{on}^{NF} = (S_t + \alpha B_t)T, \quad n_{off}^{NF} = B_t T$$

- Then test statistic is:

$$TS^{NF} = 2\left[(S_t + \alpha B_t)T \ln \frac{(1+\alpha)(S_t + \alpha B_t)T}{\alpha(S_t + (1+\alpha)B_t)T} \right. \\ \left. + B_t T \ln \frac{(1+\alpha)B_t T}{(S_t + (1+\alpha)B_t)T}\right]$$

MLE example 3:

Detectability / Sensitivity

- Weak source case: $S_t \ll \alpha B_t$

$$\sigma^{NF} = \sqrt{TS^{NF}} \approx \frac{\sqrt{T}}{\sqrt{1+\alpha}} \frac{S_t}{\sqrt{\alpha B_t}}$$

Grows as sqrt(T)

- Weak background case: $S_t \gg \alpha B_t$

$$\sigma^{NF} = \sqrt{TS^{NF}} \approx \sqrt{2S_t T \ln(1+1/\alpha)}$$

Note what happens here when $\alpha \rightarrow 0$ (which corresponds to perfectly well determined “zero” on-source background) the significance becomes infinite. If you have no background then even one event is a significant.

MLE example 3:

Detectability / Sensitivity

Minimum source strength to achieve detection at some threshold σ_{det}

- Weak source case: $S_t \ll \alpha B_t$

$$S_t > \frac{\sigma_{det}\sqrt{\alpha B_t}}{\sqrt{T}}\sqrt{1+\alpha}$$

Minimum detectable flux decreases as 1/sqrt(T) and depends on B_t : “Background-dominated regime”

- Weak background case: $S_t \gg \alpha B_t$

$$S_t > \frac{\sigma_{det}^2}{T}\frac{1}{2\sqrt{1+1/\alpha}}$$

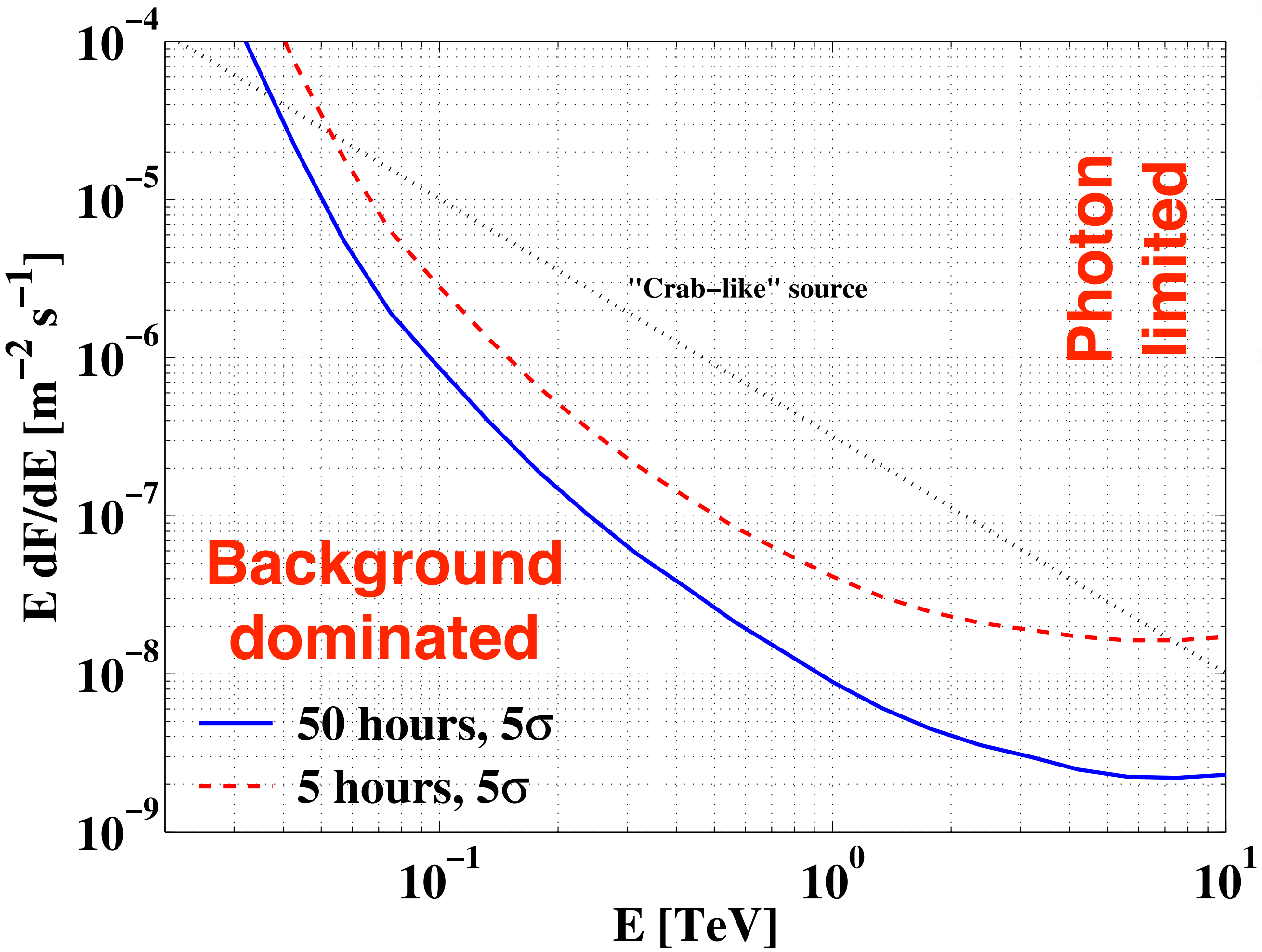
Roughly this says that the number of detected photons must be larger than σ^2 (times some constant): $S_t T = n_{det} > C\sigma_{det}^2$
eg. must detect 25 photons for 5σ.

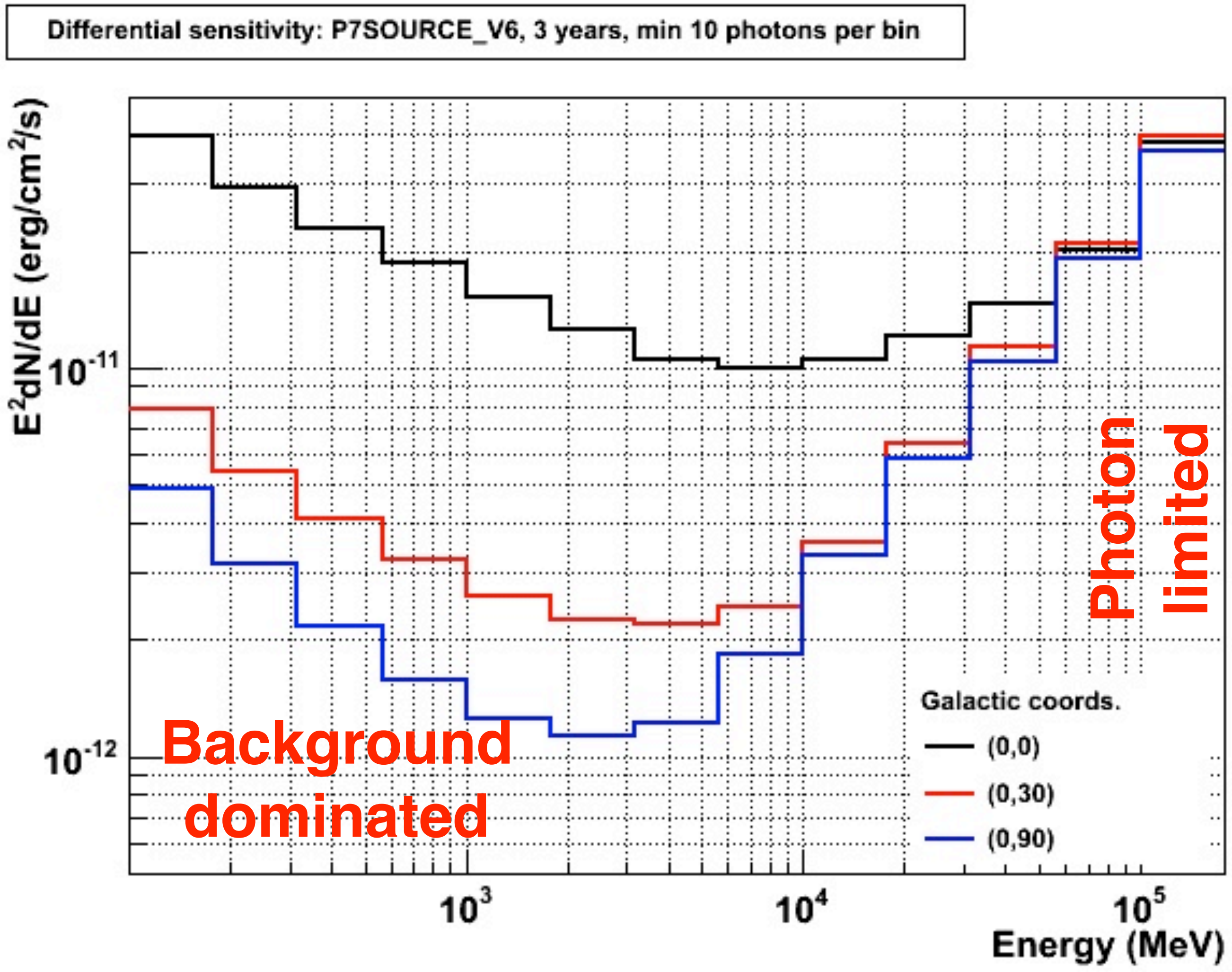
Minimum detectable flux decreases as 1/T and is independent of B_t : “Photon-limited regime”

MLE example 3:

Detectability / Sensitivity

"Differential sensitivity" plots, i.e. sensitivity in logarithmic energy bands





Sensitivity for ACT array of 4 telescopes for 5 and 50 hours of observation.

Low energies: sqrt(10) × improvement.
High energies: 10 × improvement.

LAT sensitivity from FSSC site for different background levels (Galactic or extra-galactic).

Low energies: big dependency
High energies: almost no dependency.

MLE example 3:

Systematic errors

What if assumed value of alpha is incorrect?

- Assume there is no real source:

$$n_{on}^{NF} = \alpha_t B_t T = \alpha(1+\delta)B_t T,\ n_{off}^{NF} = B_t T$$

where the error in alpha is small: $\delta \ll 1$

- Then: $\hat{S}^{NF} = B_t \alpha \delta$

$$\sigma^{NF} = \sqrt{TS^{NF}} \approx \frac{\sqrt{T}}{\sqrt{1+\alpha}} \delta \sqrt{\alpha B_t}$$

- This looks like a real signal. Accurate knowledge of experimental response is critical. MLE is only as good as the model!

Bayesian statistics

- Likelihood function has no meaning itself, e.g., it is not a probability. Its usefulness comes from theorems such as the LRT.
- MLE belongs to the class of “frequentist” statistical methods: talk about the results of repeated hypothetical experiments.
- Can produce confidence intervals where the true parameter value would lie inside the interval in a certain % of hypothetical expts.

Bayesian statistics

- In Bayesian statistics we talk about the “probability” that the parameters have certain values.
- Bayes’ theorem:

$$P(\Theta|X) = \frac{P(\Theta)P(X|\Theta)}{P(X)} \propto P(\Theta)\mathcal{L}(\Theta|X)$$

  relates probability after experiment has been done to probability before.

- Can think of this as refining our belief about the model through experimental results.

Review

- ML provides "cookbook" for estimation and hypothesis testing:
  - estimates: maximum of likelihood
  - errors: curvature of log-likelihood surface
  - TS and significance: is improvement in log-$\mathcal{L}$ over null hypothesis consistent with χ^2?
- Significance expected to grow as sqrt(T), but sensitivity can improve as 1/T if photon limited.
- Systematic errors important to consider

Onwards to LAT analysis...

- LAT ML analysis is fundamentally the same a what we have seen here (but more complex).
- Channels organized by sky position and energy (i.e. 3-dimensions). Millions of channels typical.
- Model is Poisson for each channel with mean determined by:
  - spatial-spectral model provided by user
  - observational response (calculated by software from IRFs provided by LAT team)
- MLE by software: errors, covariances, TS, etc